Thank you for your suggestion and understanding. We will include a brief theoretical analysis of the second-order temporal accuracy in the appendix.

The IMEX Runge–Kutta (IMEXRK) scheme used in this work are shown in Eq.(24)-(27):

$$\frac{d\boldsymbol{y}}{dt} = \boldsymbol{f}^{\mathrm{EX}}(\boldsymbol{y}) + \boldsymbol{f}^{\mathrm{IM}}(\boldsymbol{y}), \tag{24}$$

$$\boldsymbol{y}^{(1)} = \boldsymbol{y}^{(n)} + \Delta t \boldsymbol{f}^{\mathrm{EX}}\left(\boldsymbol{y}^{(n)}\right), \tag{25}$$

$$\boldsymbol{y}^{(2)} = \boldsymbol{y}^{(n)} + \frac{1}{2}\Delta t\left(\boldsymbol{f}^{\mathrm{EX}}\left(\boldsymbol{y}^{(n)}\right) + \boldsymbol{f}^{\mathrm{EX}}\left(\boldsymbol{y}^{(1)}\right)\right), \tag{26}$$

$$\boldsymbol{y}^{(n+1)} = \boldsymbol{y}^{(2)} + \Delta t \boldsymbol{f}^{\mathrm{IM}}\left(\boldsymbol{y}^{(n+1)}\right). \tag{27}$$

For explicit term, we first expand $\boldsymbol{f}^{\mathrm{EX}}\left(\boldsymbol{y}^{(1)}\right)$:

$$\boldsymbol{f}^{\mathrm{EX}}\left(\boldsymbol{y}^{(1)}\right) = \boldsymbol{f}^{\mathrm{EX}}\left(\boldsymbol{y}^{(n)} + \Delta t \boldsymbol{f}^{\mathrm{EX}}\left(\boldsymbol{y}^{(n)}\right)\right) = \boldsymbol{f}^{\mathrm{EX}}\left(\boldsymbol{y}^{(n)}\right) + \Delta t \boldsymbol{f}^{(\mathrm{EX})'}\left(\boldsymbol{y}^{(n)}\right)\boldsymbol{f}^{\mathrm{EX}}\left(\boldsymbol{y}^{(n)}\right) + O(\Delta t^2). \tag{B1}$$

Substituting into (26), we obtain:

$$\boldsymbol{y}^{(2)} = \boldsymbol{y}^{(n)} + \Delta t \boldsymbol{f}^{\mathrm{EX}}\left(\boldsymbol{y}^{(n)}\right) + \frac{\Delta t^2}{2}\boldsymbol{f}^{(\mathrm{EX})'}\left(\boldsymbol{y}^{(n)}\right)\boldsymbol{f}^{\mathrm{EX}}\left(\boldsymbol{y}^{(n)}\right) + O(\Delta t^3). \tag{B2}$$

The implicit term can be written via a fixed-point expansion:

$$\boldsymbol{y}^{(n+1)} = \boldsymbol{y}^{(2)} + \Delta t \boldsymbol{f}^{\mathrm{IM}}\left(\boldsymbol{y}^{(2)}\right) + O\left(\Delta t^2\right). \tag{B3}$$

Substituting (B2) into (B3), we obtain:

$$\begin{aligned}\boldsymbol{y}^{(n+1)} &= \boldsymbol{y}^{(n)} + \Delta t \boldsymbol{f}^{\mathrm{EX}}\left(\boldsymbol{y}^{(n)}\right) + \frac{\Delta t^2}{2}\boldsymbol{f}^{(\mathrm{EX})'}\left(\boldsymbol{y}^{(n)}\right)\boldsymbol{f}^{\mathrm{EX}}\left(\boldsymbol{y}^{(n)}\right) + O(\Delta t^3) \\ &\quad + \Delta t \boldsymbol{f}^{\mathrm{IM}}\left(\boldsymbol{y}^{(2)}\right) + O\left(\Delta t^2\right).\end{aligned} \tag{B4}$$

Then, we expand $\boldsymbol{f}^{\mathrm{IM}}\left(\boldsymbol{y}^{(2)}\right)$ and it:

$$\boldsymbol{f}^{\mathrm{IM}}\left(\boldsymbol{y}^{(2)}\right) = \boldsymbol{f}^{\mathrm{IM}}\left(\boldsymbol{y}^{(n)}\right) + \frac{\Delta t}{2}\boldsymbol{f}^{(\mathrm{IM})'}\left(\boldsymbol{y}^{(n)}\right)\boldsymbol{f}^{\mathrm{IM}}\left(\boldsymbol{y}^{(n)}\right) + O(\Delta t^2). \tag{B5}$$

We can find that $\boldsymbol{f}^{\mathrm{IM}}\left(\boldsymbol{y}^{(2)}\right)$ is first-order and $\Delta t \boldsymbol{f}^{\mathrm{IM}}\left(\boldsymbol{y}^{(2)}\right)$ is second-order. Substituting (B5) into (B4), we obtain:

$$\begin{aligned}\boldsymbol{y}^{(n+1)} &= \boldsymbol{y}^{(n)} + \Delta t \boldsymbol{f}^{\mathrm{EX}}\left(\boldsymbol{y}^{(n)}\right) + \frac{\Delta t^2}{2}\boldsymbol{f}^{(\mathrm{EX})'}\left(\boldsymbol{y}^{(n)}\right)\boldsymbol{f}^{\mathrm{EX}}\left(\boldsymbol{y}^{(n)}\right) + O(\Delta t^3) \\ &\quad + \Delta t\left(\boldsymbol{f}^{\mathrm{IM}}\left(\boldsymbol{y}^{(n)}\right) + \frac{\Delta t}{2}\boldsymbol{f}^{(\mathrm{IM})'}\left(\boldsymbol{y}^{(n)}\right)\boldsymbol{f}^{\mathrm{IM}}\left(\boldsymbol{y}^{(n)}\right)\right) + O(\Delta t^3).\end{aligned} \tag{B6}$$

By rearranging the terms, we obtain (B7), which matches the Taylor expansion of the exact solution to second order.

$$
\begin{aligned}
\boldsymbol{y}^{(n+1)} &= \boldsymbol{y}^{(n)} + \Delta\left(t\boldsymbol{f}^{\mathrm{EX}}\left(\boldsymbol{y}^{(n)}\right) + \boldsymbol{f}^{\mathrm{IM}}\left(\boldsymbol{y}^{(n)}\right)\right) \\
&\quad + \frac{\Delta t^2}{2}\left(\boldsymbol{f}^{(\mathrm{EX})'}\left(\boldsymbol{y}^{(n)}\right)\boldsymbol{f}^{\mathrm{EX}}\left(\boldsymbol{y}^{(n)}\right) + \boldsymbol{f}^{(\mathrm{IM})'}\left(\boldsymbol{y}^{(n)}\right)\boldsymbol{f}^{\mathrm{IM}}\left(\boldsymbol{y}^{(n)}\right)\right) + O(\Delta t^3).
\end{aligned} \tag{B7}
$$

This Hence, the scheme is formally second-order accurate in time, despite the use of a first-order implicit method.