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5th July 2024
Dear editor and reviewer,

Re:” **Three-Dimensional Analytical Solution of Self-potential from Regularly Polarized Bodies in Layered Seafloor Model**”

Thank you for taking time to review our paper in such detail. We appreciate the chance to revise this paper.
Through your review, we have recognized numerous deficiencies in the manuscript. In response to the reviewers' comments, we have made extensive revisions to the manuscript. We have supplemented the content of the formula derivations, adding more detailed steps to make them easier to understand. In Section 2.1, we demonstrated the equivalence of the self-potential generated by a uniformly polarized sphere and that generated by an electric dipole. Subsequently, we derived the three-dimensional analytical solutions of self-potential in two-layer and three-layer media. In Section 3, we revised all the figures, using new three-dimensional potential distribution plots to better present the results of the analytical solutions. We also added substantial explanatory content to improve the readability of the article. In Section 4, we provided a more detailed description of the experimental process. Recognizing the limitations of previously comparing the central survey line results, we have now compared the experimental results with the two-dimensional slices of the analytical solutions and evaluated them by calculating the coefficient of determination(R^2). Additionally, we corrected erroneous descriptions in the manuscript and standardized the notation, including symbols and Greek letters.
In this response letter, your comments in blue and our response in black. Because we have made extensive revisions, it may not be possible to showcase all the changes through the discussion alone. Therefore, we have attached the revised manuscript.

This paper proposed a method to solve the analytical solution of the SP and based on the mirror image current theory, the 2D and 3D analytical solution formulas were derived. Some things in the paper were not clear and properly stated which made the paper difficult to read especially the derivation of the formulas which is the major goal of the paper. Below are some of my observations

1.I thought the formula accords with the Laplace equation in spherical coordinates is given by $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)=0$ What happened to the denominators (r2) in your own quoted equation?

Thank you for your insightful comment regarding the presence of $1/r^2$ terms in the Laplace equation in spherical coordinates. I would like to clarify that the formula in question is indeed derived within the context of spherical coordinates. In a uniformly polarized sphere, the external potential U is distributed symmetrically about the polarization axis and is independent of the azimuthal angle. This specific symmetry leads to a simplified form of the Laplace equation that we have used in our derivations.

he Laplace equation in spherical coordinates (r,θ,ϕ) is given by:

$$\nabla^2 U=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2}=0$$

Given the problem's symmetry, the potential U is independent of the azimuthal angle ϕ. Thus, the equation reduces to:

$$\nabla^2 U=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right)=0$$

In our derivation, we focus on solving the radial part of the equation:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)=0$$

This can be rewritten as:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)=0$$

Integrating this equation with respect to r:

$$r^2\frac{\partial U}{\partial r}=A$$

where A is an integration constant. Solving for U:

$$U(r)=-\frac{A}{r}+B$$

where B is another integration constant.

When we present the final expression for the potential U(r), the term A/r results from the integration process. The $1/r^2$ term is inherent in the integration steps and thus does not explicitly appear in the final potential expression.

This form of the equation aligns with the theoretical framework described in our paper. Similar derivations and explanations can be found in classical works on potential theory, such as those by MacMillan (1958) in "The Theory of the Potential".

I hope this explanation clarifies the basis of our derivations and demonstrates that the formulas presented are consistent with the Laplace equation in spherical coordinates. We will incorporate additional explanations in the revised manuscript to clarify the derivation steps and the context in which the $1/r^2$ terms are inherently accounted for due to the spherical symmetry of the problem.

2.Was equation 3 (the general solution of potential) in section 2.1 something you guys came up with, or was it already there? If the equation is already known, you ought to credit the authors or the source, in my opinion. Although equations 1.6 and 1.7 were mentioned, your manuscript does not contain any equations of that kind.

Equation 3 was derived using the method of separation of variables. Here, we have provided the complete solution process.

The Laplace equation in spherical coordinates is given by:

$$\frac{\partial}{\partial R}\left(R^2\frac{\partial U}{\partial R}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right)=0$$

We want to find a solution $U(R,\theta)$ in the form:

$$U(R,\theta)=R^nP_n(\cos\theta)$$

where $P_n(\cos\theta)$) are the Legendre polynomials.

Assume the solution can be separated into a radial part $R(R)$ and an angular part $\Theta(\theta)$:

$$U(R,\theta)=R(R)\Theta(\theta)$$

Substitute this into the Laplace equation:

$$\frac{\partial}{\partial R}\left(R^2\frac{\partial(R(R)\Theta(\theta))}{\partial R}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial(R(R)\Theta(\theta))}{\partial\theta}\right)=0$$

Expand the equation:

$$\Theta(\theta)\frac{\partial}{\partial R}\left(R^2\frac{\partial R(R)}{\partial R}\right)+R(R)\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right)=0$$

Separate the equation into two independent parts:

$$\frac{1}{R(R)}\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{dR(R)}{dR}\right)=-\frac{1}{\Theta(\theta)}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)=\lambda$$

where λ is the separation constant.

The angular part of the equation is:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)+\lambda\Theta(\theta)=0$$

The solution to this equation is the Legendre polynomial:

$$\Theta(\theta)=P_n(\cos\theta)$$

where λ=n(n+1) and n is an integer.

The radial part of the equation is:

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{dR(R)}{dR}\right)-\lambda R(R)=0$$

Substitute λ=n(n+1):

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{dR(R)}{dR}\right)-n(n+1)R(R)=0$$

Assume the solution is of the form:

$$R(R)=A_nR^n+\frac{B_n}{R^{n+1}}$$

Combine the radial and angular parts to get the total solution:

$$U(R,\theta)=\sum_{n=0}^{\infty}\left(A_nR^n+\frac{B_n}{R^{n+1}}\right)P_n(\cos\theta)$$

We apologize for our oversight; Equations (1.6) and (1.7) do not exist. It should refer to Equations (9) and (10). We have made the corrections in the main text.

3.What steps led from equation 7 to equation 8? Is it feasible to simplify it and the connectivity for comprehension? If you have derived the majority of these equations, please state so. If not, provide appropriate citation by quoting the academics.

The transition from equation 7 to equation 8 (equation 10 and equation 11 in the revised manuscript) involves the principle of superposition of potentials, which is fundamental in electrostatics and electromagnetic theory. Specifically, the formula provided for Φ_{sea} represents the combined potential due to a source dipole and its image dipole.

The scalar potential U caused by a constant electric dipole is given by:

$$U=\frac{Idl}{4\pi\sigma}\left(\frac{-x}{R^3}\right)=-P_0\frac{x}{R^3}$$

where *I* is the current, *dl* is the length element, σ is the conductivity, and R is the distance from the dipole to the observation point.

The potential in a medium with a boundary (such as the sea-air interface) can be determined by considering the contribution from both the real dipole and its image dipole. This method ensures the boundary conditions are satisfied.

The potential in the seawater (z>0) is given by the sum of the potentials due to the real dipole(P=I_xdl)and the image dipole(P'=I_x'dl):

$$\Phi_{sea} = \frac{I_x dl(x - x_0)}{4\pi\sigma_1 R_1^3} + \frac{I_x^{'} dl(x - x_0)}{4\pi\sigma_1 R_0^3} \quad (z > 0)$$

R_1 and R_0 are the distances from the observation point to the real and image dipoles.

4.For someone who is not in the geosciences, most of the derivations might be unclear for them. I believe you should correctly demonstrate the connections and how you arrived to equation 13. All I'm asking is that you provide a step-by-step explanation of the derivations as this is your new formula, so that other researchers can comprehend the equations and draw conclusions from them. The Table 1 formulas must to be clearly described or demonstrate how you arrived at each of their instances.

Thanks for your good suggestion. The image method is a mathematical technique used to simplify boundary value problems in electrostatics. It involves replacing the boundary with an equivalent charge distribution (the image charge or dipole) that ensures the boundary conditions are satisfied. In the paper, we first discuss 2-layer models. Whether it is the seawater-air model or the seawater-seafloor model, the essence of the image method is to replace the boundary with an electric dipole or a virtual image dipole, thereby simplifying the "one source with two mediums" problem to a "one medium with one source and one image source" problem. This is done while satisfying the actual boundary conditions. When the measuring points and the source are in the same medium, the potential calculation must take into account the conductivity of the medium. Therefore, the potential caused by a constant electric dipole is the superposition of the potentials generated by the source point and the image point (located in the other medium). When the measuring point and the source point are not in the same medium, to satisfy the boundary conditions:

1）The potential must be continuous across the boundary. This means that the potential just above the boundary in the air must equal the potential just below the boundary in the seawater

2）The normal component of the current density (or electric field) should also be continuous across the boundary. This condition ensures that the boundary behaves correctly according to Maxwell's equations.

the image dipole coincides with the real dipole. The combined dipole moment is used to calculate the potential in the air. Because the dipole moment of the image dipole is unknown, we need to solve it using the boundary conditions.

In the seawater-air model, the image dipole moments for the two different measurement scenarios are solved as shown in Equations (14) and (15). In the seawater-seafloor model, we can use the same method to solve for the dipole moments in different situations by combining the boundary conditions. When the measuring point is located in seawater:

$$I_x dl''' = \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2} I_x dl$$

And when the measuring point is located in seafloor,

$$Ixdl'''' = \frac{2\sigma_2}{\sigma_1 + \sigma_2} I_x dl$$

The following diagram is a schematic representation of the two-layer medium after processing using the image method.

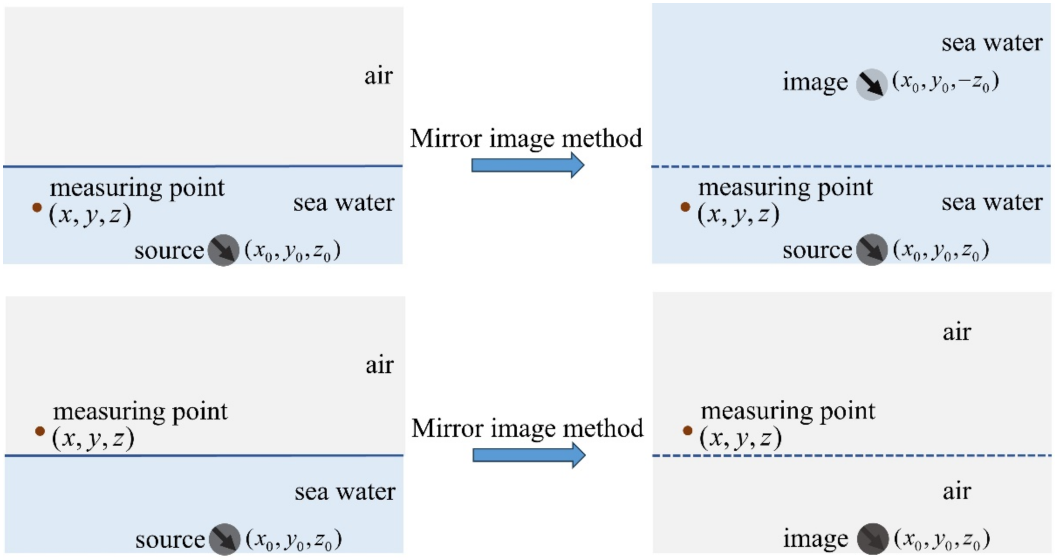

Figure 1. Schematic representation of the two-layer medium after processing using the image method.

In the three-layer model, the image method generates an infinite number of images. As an example, we consider the first three images (as shown in Figure 2). The source point generates corresponding images in the other two media, referred to as the first image. The generated images in turn create new images in the other medium. For instance, an image dipole 1 generated in the air by the source point will produce a second image dipole 1-1 in the seafloor medium; similarly, an image dipole 2 generated in the seafloor medium will produce another second image dipole 2-2 in the air. This process continues, generating an infinite number of image dipoles. We calculate the potential generated by each image point in a manner similar to the two-layer model, based on the boundary conditions. Upon solving for different image points, we find that the image points can be categorized into four types, based on the depth patterns of the image dipoles. The coordinates of the first type of image dipole are $(x_0, y_0, 2\mathrm{m}D - z_0)$, $\mathrm{m} = 1, 2, \ldots$, with the corresponding dipole moment solved as $\left(\frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2}\right)^{\mathrm{m}} \boldsymbol{I}_x \boldsymbol{dl} = \eta^m \boldsymbol{I}_x \boldsymbol{dli}, m = 1, 2, \ldots$. The vector distance between the measuring point and each image point can be expressed as: $\boldsymbol{r}_{1m} = (x - x_0)\boldsymbol{i} + (y - y_0)\boldsymbol{j} + (z - 2mD + z_0)\boldsymbol{k}$. The image 2 and image 2-2-2 in the figure represents this type of image. The second type of image dipole located at $(x_0, y_0, 2\mathrm{m}D + z_0), m = 1,2,...$, with the corresponding dipole moment solved as $\left(\frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2}\right)^{\mathrm{m}} \boldsymbol{I}_x d\boldsymbol{l} = \eta^m \boldsymbol{I}_x dl\boldsymbol{i}, m = 1,2,...$. The vector distance between the measuring point and each image point can be expressed as: $\boldsymbol{r}_{2m} = (x - x_0)\boldsymbol{i} + (y - y_0)\boldsymbol{j} + (z - 2mD - z_0)\boldsymbol{k}$. The image 1-1 represents this type. The third type of image dipole located at $(x_0, y_0, -2\mathrm{n}D + z_0), n = 0,1,...$, with the corresponding dipole moment $\left(\frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2}\right)^{\mathrm{n}} \boldsymbol{I}_x d\boldsymbol{l} = \eta^n \boldsymbol{I}_x dl\boldsymbol{i}, n = 0,1,...$. The vector distance between the measuring point and each image point can be expressed as: $\boldsymbol{r}_{1\mathrm{n}} = (x - x_0)\boldsymbol{i} + (y - y_0)\boldsymbol{j} + (z + 2nD - z_0)\boldsymbol{k}$. The source and image2-2 represent this type. The coordinates of the first type of image dipole are $(x_0, y_0, -2\mathrm{n}D - z_0), n = 0,1,...$, with the corresponding dipole moment $\left(\frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2}\right)^{\mathrm{n}} \boldsymbol{I}_x d\boldsymbol{l} = \eta^n \boldsymbol{I}_x dl\boldsymbol{i}, n = 0,1,...$. The vector distance between the measuring point and each image point can be expressed as: $\boldsymbol{r}_{2\mathrm{n}} = (x - x_0)\boldsymbol{i} + (y - y_0)\boldsymbol{j} + (z + 2nD + z_0)\boldsymbol{k}$. The image 1 and image 1-1-1 represent this type of image. The classification of these image points is based on their positions. We can see that, apart from the source point, the third and fourth types of image points are located in the air, while the first and second types are located in the seafloor. Other than this, there are no other physical differences between the various types of image points.

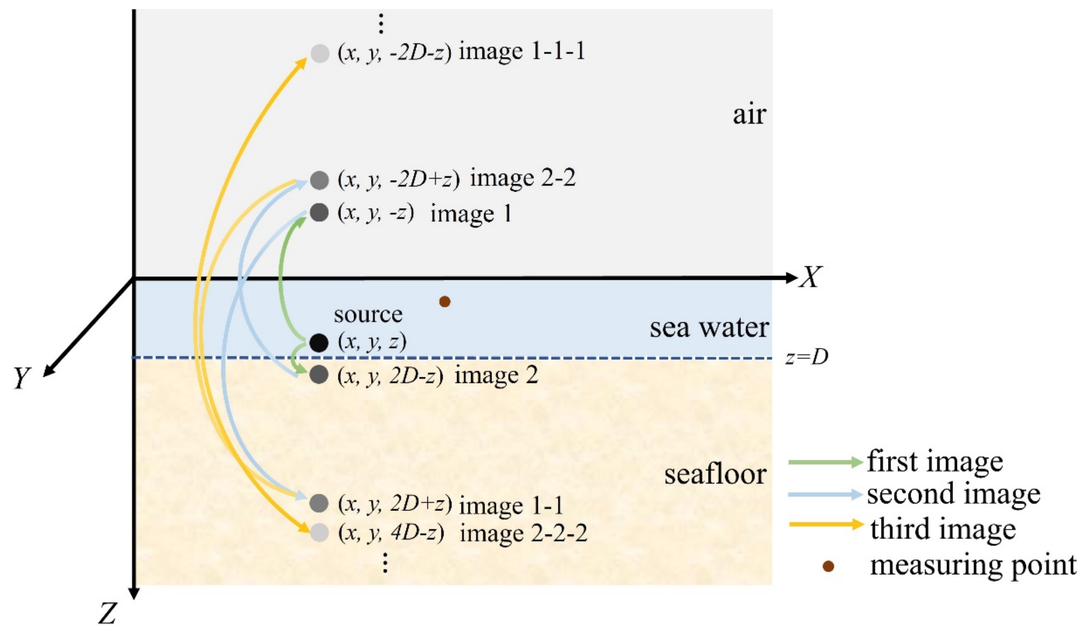

Figure 2. Schematic representation of the three-layer medium after processing using the image method (Using the first three image points as an example).

5.Is it possible to put the result analysis of Section 3 and Section 4 in table format to support the figure results? The table will help to make the analysis more comprehensible.

We understand the importance of presenting results in a clear and comprehensible manner. In our revised manuscript, we have provided detailed analyses in Section 3 and Section 4 using three-dimensional potential distributions and depth slices, as well as calculating the R^2 values in section 4. These visual representations are crucial for illustrating the spatial variations and correlations inherent in our study. We believe that the three-dimensional figures and depth slices offer a more intuitive understanding of the spatial distribution of self-potential signals, which might be challenging to convey effectively in a tabular format. Additionally, the R^2 values calculated and presented in Section 4 quantitatively support the accuracy and correlation of our model predictions with the experimental data. We hope this explanation clarifies our approach and the rationale behind our presentation choices.

6.For better understanding, it would be beneficial if you could explain section 3 following figure 4, and each figure should have an explanation. Section 4 should follow the same procedure as related to Figure 5. Provide a thorough explanation for each of the following three figures: a horizontal electric dipole, b vertical electric dipole, and c tilted electric dipole.

In Section 3, we have supplemented extensive explanatory content, including the interpretation of self-potential distributions generated by horizontal, vertical, and inclined dipoles. We have conducted a detailed analysis of self-potential slices at different depths. In Section 4, we have added descriptions of the experimental setup and procedures, and we have compared and evaluated the measured self-potential results with the forward modeling results.

7.It is not clear if you compared the formula for the 2D analytical solution you generated with the 2D measured data, just as you did for the 3D analytical solution.

We thank you for your insightful suggestion. The primary focus of our research is on three-dimensional (3D) analysis, which is why we developed and presented the 3D analytical solution. Correspondingly, we conducted 3D physical simulations to validate our analytical models. Our experimental setup is inherently three-dimensional, with the entire experiment conducted in a 3D space where self-potential can be recorded at any point within this space.

We realize that our initial description might have been unclear and potentially confusing. To address this, we have revised the relevant sections of the manuscript to better emphasize our 3D approach and its importance to our study.

The revisions clarify that the core objective was to establish and validate a comprehensive 3D analytical solution. The physical simulations were also designed in a 3D context to provide accurate validation of these models.

We hope this detailed explanation clarifies our approach and addresses your concern. Thank you for your constructive feedback, which has been instrumental in refining our study.

However, to make it easier, it would be even better if the previously mentioned points—particularly the derivation formula and the results—were broken down and discussed in detail while keeping in mind potential readers of the work who are not geoscientists. The current state of the document may make it somewhat difficult for someone who is not in the field of geoscience to understand, and the goal of any research paper is to provide clarity so that others can benefit from your work.