Overview

This work presents two generalized hybrid methods for atmospheric inversion modeling schemes, one involving a fixed prior mean and another with an unknown prior mean. These methods offer greater computational efficiency during the inversion process and approximate posterior covariance matrices that may otherwise be unfeasible to calculate. Results were presented in the context of a continental inversion scheme using synthetic CO2 observations from the OCO-2 instrument. Although this manuscript is mathematically dense, it is well written and informative. The given detail of the Bayesian inversion scheme may narrow the readership of this article to those specializing in this type of mathematics while being more challenging to implement by those who have only an operational knowledge of inversions. Nonetheless, this work provides a timely improvement to the atmospheric inversion modeling process as the number of space-based CO2 observing platforms is set to increase. I recommend that this article be published once the following comments have been addressed.

General Comments

The introduction is clear and well written; however, additional motivation would strengthen this work. Currently, the introduction poses the content as an "interesting math problem" but fails to answer the question of *why are inversions so important that they need to be done faster?* Since this publication was submitted to GMD, the physical implications of this work should also be mentioned. Observation systems are increasing in number, allowing scientists to better constrain anthropogenic influences on the climate. To mitigate these influences, rapid monitoring, reporting, and verification of emissions (anthropogenic and otherwise) is needed to ensure cities, regions, and nations are working to reduce them. Ultimately, the work presented in this manuscript will assist with this challenge, making it more than just "interesting math". These points should be reiterated in the conclusion.

In line 22, it is stated that "the number of greenhouse gases and air pollution measurements has greatly expanded in the past decade, enabling investigations of surface fluxes across larger regions, longer time periods, and/or at finer spatial and temporal detail." However, the only example of a ground-based network given was NOAA's GML. Several other local/regional ground-based monitoring systems exist and could be offered to readers as additional examples. Salt Lake City's UUCON, Indianapolis INFLUX, UC Berkeley's BEACON. (Some of these networks may provide data to NOAA's GML data archive but their local/regional focus is worth noting.)

Approximate posterior covariance matrices, $\widetilde{\boldsymbol{Q}}_{\text{post}}$ and $\tilde{\boldsymbol{\Gamma}}_{\text{post}}$ are given in line 264 and Equation B6 respectively. For the effectiveness of these approximation methods, the reader is referred to a citation: Saibaba et al., 2020. A few sentences within this manuscript describing results from Saibaba et al., the effectiveness of the approximations, limitations, etc. would benefit the curious reader.

The two case studies reported in this work made use of pseudo-observations from NASA's OCO-2 instrument to estimate CO2 fluxes at 3-hour intervals and 1º x 1º spatial resolution. It is unclear from the text alone how OCO-2 soundings were incorporated into the inversion scheme. Generally, OCO-2 soundings are densely spaced (~2-3km apart) and demonstrate varying spatial correlation in error. How is the assumption of spatially independent errors ($R = \sigma^2 \boldsymbol{I}$) justified?

How/if soundings were spatiotemporally aggregated for this study is not clear. These details should be *briefly* included in the text while referring readers to other sources for *more* detail.

Figures 4 and 7 present the results of this work in a concise way; however, it is difficult to compare the effectiveness of so many different methods. Consider plotting the *differences* between the estimated and true fluxes. Obviously, the goal is to get as close to the true flux as possible. So, using a blue (-) to red (+) color gradient will easily show *where* the inversion is overestimating, underestimating, and effectively reproducing true emissions. Comparing gradients associated with relative differences may be easier across the various model outputs.

Technical Specifics

Moving the citations in line 28 to immediately follow its associated observing system instead of listing them at the end of the statement would be helpful for readers.

In line 82, it may be worth pointing out, for readers unfamiliar with this technique, that flux values from spatially explicit 2D arrays (x,y) are represented as a vector in this technique. Thus, *n* is the number of cells in the domain of interest. i.e. – enforce how $\vec{s}$ is constructed.

Minor point: In lines 52, 139, 150, 157, 247, 254, 265, 370, papers are referenced directly in the text but they still have parentheses around them. Typically, parentheses are only included if a statement is cited without direct reference in the text. (Is this GMD formatting?)

On line 247, the sentence beginning with "Instead, we follow the approach described in…" the word 'using' is included twice, making the sentence awkward to read.

In line 291, *p* appears to be "filler" dimensions in the matrix to ensure that matrix multiplications work out. Although this can be intuited from the mathematics, it would be beneficial to point it out in the text.

In line 343, $\sigma = 2$. It is suggested in the text that this corresponds to nlevel 100%. So, how can nlevel 50% be 0.5648 as in Table 1? Isn't the percentage of nlevel relative to this value from Miller et al., 2020? This is unclear in the text.

What are the units of θ_t and θ_g in line 370?

Apparent typo in line 373: "… sparse can…" should be "… sparse <u>and</u> can…"

Figure #3 needs some work. The axes and title texts need to be made smaller to better fit the in-plot labels of $\lambda\sigma$. Smaller text will allow for bigger plot areas.