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Interactive comment on "Quasi-hydrostatic equations for climate models and the study on linear instability" *by* Robert Nigmatulin and Xiulin Xu

Anonymous Referee #2

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This paper tries to analyse wavelike perturbations from a quasi-hydrostatic equation set. This equation set is put into a framework in z-coordinates. This lets the equations look like as they are formulated as we are used to them in p-coordinates (hence uses some kind of the Richardson equation). The paper seeks for potential unstable growing solutions and finds them. However, in my opinion, the results are questionable, because they are not leading to the results which are usually obtained, namely the solution of gravity waves and their dispersion relation. And as we know, GW solutions are not growing in time, but are pure waves.

My general impression is that since the underlying physics of the system does not

differ from other analyses to be be found in the literature, the same known results should come to the fore. A dispersion analysis of waves in a system should not depend on the specific prognosic variables or differently written equations. All those systems must lead to the same results, but equation (4.3) does not coincide with other known solutions.

I tried to figure out where the problem might be in the actual derivation. Two points are somehow strange to me. First, the system (2.23) consists of 6 instead of 5 equations. So, there are some linear dependencies among the equations. But, this might be not so problematic, since the found dispersion relations finds an equation in ω^3, which means three solutions, which is fine. Second, compared to more traditional approaches, the step of the Bretherton (1966) transfromation has not been done. Consequently the coefficient matrices do not have constant coefficients (appendix A), because the density depends on height. I do not know, which consequences arise due to this missing step.

In the following, I copy the part of my lecture of gravity waves, which focuses on the derivation of them under the hydrostatic constraint (omission of blue terms) and under shedding of acoustic waves (omission of red terms). Perhaps the authors could figure out, how their derivation differs from these conventional steps and how their approach could be brought under the umbrella of known results.

Here follows the lecture part:

1 Gravity waves (GWs)

1.1 Dispersion relation for gravity waves

- background state assumes arbitrarily constant N^2 (includes the special isothermal case $N^2_{iso} = g^2/(c_pT)$)

- hydrostatic approximation is not needed
- assuming incompressibility is not needed, acoustic waves are later separated in the dispersion relation itself
- it becomes obvious that the amplitudes of the wave perturbations have exponential behavior with height
- a constant Coriolis parameter f is assumed
- for simplicity we assume a dry atmosphere
- irreversible processes like friction or heating are not included in the wave analysis (later we will include them)

The governing equations are linearized around a hydrostatic state and a mean zonal current U, and the individual time derivative is abbreviated with $d_t\cdot = \partial_t \cdot + U\partial_x\cdot$. The vertical advection is separated out from this operator. We have

$$u = U + u', \qquad v = v', \qquad w = w', \qquad p = p_0(z) + p', \qquad \theta = \theta_0(z) + \theta' \tag{1}$$

The equation of state is assumed for the mean state / the background separately: $p_0(z) = \varrho_0(z)RT_0$. The hydrostatic relation hold for the background state $\partial p_0/\partial z = -\varrho_0(z)g$. And we have $T_0(z) = \theta_0(z)\Pi_0(z)$.
As the thermodynamic equation we use the potential temperature equation which reads

$$d_t\theta' + w'\partial_z\theta_0(z) = 0 \qquad | \cdot g/\theta_0(z) \tag{2}$$
$$d_tb' + w'N^2 = 0 \tag{3}$$

where $b' = g\theta'/\theta$ is called the buoyancy. Brunt-Vaisala frequency / buoyancy frequency: $N^2 = g\partial_z \ln\theta_0(z)$.

The continuity equation is not used directly, but rather, an equation for the pressure perturbation is chosen as prognostic variable. Then we have as the not yet linearized equation

$$\varrho c_v d_t T + p\nabla\cdot\mathbf{v} = 0 \tag{4}$$
$$\varrho c_v d_t \frac{p}{R\varrho} + p\nabla\cdot\mathbf{v} = 0 \tag{5}$$
$$\frac{\varrho c_v}{R\varrho} d_t p - \varrho c_v \frac{p}{R\varrho^2} d_t\varrho + p\nabla\cdot\mathbf{v} = 0 \tag{6}$$
$$\frac{c_v}{R} d_t p + \frac{\varrho^2 p c_v}{R\varrho^2}\nabla\cdot\mathbf{v} + p\nabla\cdot\mathbf{v} = 0 \tag{7}$$
$$\frac{c_v}{R} d_t p + \frac{(c_v + R)p}{R}\nabla\cdot\mathbf{v} = 0 \tag{8}$$
$$\frac{c_v}{c_pRT}\frac{1}{\varrho} d_t p + \nabla\cdot\mathbf{v} = 0 \tag{9}$$
$$\frac{1}{c_s^2\varrho} d_t p + \nabla\cdot\mathbf{v} = 0 \tag{10}$$

Speed of sound: $c_s^2 = c_pRT/c_v$. Note that the vertical advection of $p_0(z)$ remains relevant when linearizing.

Original linearized equations are

$$d_tu' - fv' + \frac{1}{\varrho_0(z)}\partial_x p' = 0 \tag{11}$$
$$d_tv' + fu' + \frac{1}{\varrho_0(z)}\partial_y p' = 0 \tag{12}$$
$$d_tw' - b' + \frac{g}{c_s^2}\frac{p'}{\varrho_0(z)} + \frac{1}{\varrho_0(z)}\partial_z p' = 0 \tag{13}$$
$$d_tb' + w'N^2 = 0 \tag{14}$$

$$\frac{1}{c_s^2 \varrho_0(z)} d_t p' - \frac{g}{c_s^2} w' + \nabla \cdot \mathbf{v}' = 0 \quad (15)$$

The blue term vanishes for the hydrostatic constraint. The red term in (15) vanishes for incompressibility. In (13) the red term vanishes if the pseudo-density $\varrho^* = \rho_0(z)\theta_0(z)/\theta$ is used in the pressure gradient term. These conditions help filtering out sound waves. We must now erase the height dependency of the density. Key tool is the Bretherton transformation[1]. Define transformation: $(u'', v'', w'', b'') = \sqrt{\varrho_0(z)/\varrho_{surf}}(u', v', w', b')$ and $p'' = \sqrt{\varrho_{surf}/\varrho_0(z)}\ p'$. This transformation guarantees the exponential increase of wave amplitudes with height due to the density decrease (see earlier chapter on Rossby waves).

After transformation the system becomes

$$d_t u'' - f v'' + \frac{\partial_x p''}{\varrho_{surf}} = 0 \quad (16)$$

$$d_t v'' + f u'' + \frac{\partial_y p''}{\varrho_{surf}} = 0 \quad (17)$$

$$d_t w'' - b'' + \left(\frac{g}{c_s^2} - \frac{1}{2H}\right)\frac{p''}{\varrho_{surf}} + \frac{\partial_z p''}{\varrho_{surf}} = 0 \quad (18)$$

$$d_t b'' + w'' N^2 = 0 \quad (19)$$

$$\frac{d_t p''}{c_s^2 \varrho_{surf}} + \nabla_h \cdot \mathbf{v}_h'' + \left(\frac{1}{2H} - \frac{g}{c_s^2}\right) w'' + \partial_z w'' = 0 \quad (20)$$

Scale height:

$$\frac{1}{H} = -\frac{1}{\varrho_0}\frac{\partial \varrho_0}{\partial z} = \frac{N^2}{g} + \frac{g}{c_s^2} \quad (21)$$

[1]Bretherton FP. 1966. The propagation of groups of internal gravity waves in a shear flow. QJRMS 92: 466-480.

Wave ansatz for an arbitrary transformed variable: $\psi'' = A_\psi \exp(i(kx + ly + mz - \omega t))$
Intrinsic frequency: $\omega_I = \omega - kU$
Linear equation system:

$$\begin{pmatrix} -i\omega_I & -f & 0 & 0 & ik \\ f & -i\omega_I & 0 & 0 & il \\ 0 & 0 & -i\omega_I & -1 & im - \frac{1}{2H} + \frac{g}{c_s^2} \\ 0 & 0 & N^2 & -i\omega_I & 0 \\ ik & il & im + \frac{1}{2H} - \frac{g}{c_s^2} & 0 & \frac{-i\omega_I}{c_s^2} \end{pmatrix} \cdot \begin{pmatrix} A_u \\ A_v \\ A_w \\ A_b \\ A_{p/\varrho_{surf}} \end{pmatrix} = 0 \quad (22)$$

det (...) = 0 defines the dispersion relation

$$\begin{aligned} \det(\ldots) &= -i\omega_I \left\{ -i\omega_I \left(i\omega_I \omega_I \frac{\omega_I}{c_s^2} + i\omega_I \left(im + \frac{1}{2H} - \frac{g}{c_s^2} \right) \left(im - \frac{1}{2H} + \frac{g}{c_s^2} \right) - \frac{i\omega_I}{c_s^2} N^2 \right) \right. \\ &\quad \left. -il \left(-il\omega_I \omega_I + ilN^2 \right) \right\} \\ &\quad + f \left\{ f \left(i\omega_I \omega_I \frac{\omega_I}{c_s^2} + i\omega_I \left(im + \frac{1}{2H} - \frac{g}{c_s^2} \right) \left(im - \frac{1}{2H} + \frac{g}{c_s^2} \right) - \frac{i\omega_I}{c_s^2} N^2 \right) \right. \\ &\quad \left. -il \left(-ik\omega_I \omega_I + ikN^2 \right) \right\} \\ &\quad + ik \left\{ f \left(-il\omega_I \omega_I + ilN^2 \right) + i\omega_I(-ik\omega_I \omega_I + ikN^2) \right\} \quad (23) \\ &= 0 \end{aligned}$$

This gives

$$\begin{aligned} 0 &= (f^2 - \omega_I^2) \left(i\omega_I \omega_I \frac{\omega_I}{c_s^2} + i\omega_I (im + \frac{1}{2H} - \frac{g}{c_s^2})(im - \frac{1}{2H} + \frac{g}{c_s^2}) - \frac{i\omega_I}{c_s^2} N^2 \right) \quad (24) \\ &\quad -il^2 \omega_I(-\omega_I \omega_I + N^2) + lkf(-\omega_I \omega_I + N^2) - klf(-\omega_I \omega_I + N^2) - ik^2 \omega_I(-\omega_I \omega_I + N^2) \end{aligned}$$

And shorter

$$0 = [f^2 - \omega_I^2] \left[\frac{\omega_I}{c_s^2}(\omega_I \omega_I - N^2) + \omega_I \left[-m^2 - \left[\frac{1}{2H} - \frac{g}{c_s^2} \right]^2 \right] \right] + \omega_I (l^2 + k^2)(\omega_I \omega_I - N^2) \quad (25)$$

$$0 = \frac{\omega_I}{c_s^2}(\omega_I \omega_I - N^2)(f^2 - \omega_I^2) + \omega_I \left[(\omega_I^2 - f^2) \left[m^2 + \left[\frac{1}{2H} - \frac{g}{c_s^2} \right]^2 \right] + (l^2 + k^2)(\omega_I \omega_I - N^2) \right] \quad (26)$$

The red terms are significant for acoustic waves. For $g = 0$, $N^2 = 0$ and $f^2 = 0$ holds: $\omega_{I,ac}^2 = c_s^2(k^2 + l^2 + m^2)$.
A stationary solution, the Rossby mode, $\omega_I = 0$, exists in any case, because $\beta = 0$.
GWs are derived by neglecting all the red terms, hence acoustic waves are filtered out.
Hydrostatic GWs disregard the blue term. This gives

$$\omega_I^2 = f^2 + \frac{N^2(k^2 + l^2)}{m^2 + \frac{1}{4H^2}} \quad (27)$$

General GWs have the dispersion relation

$$\omega_I^2 = \frac{f^2(m^2 + \frac{1}{4H^2}) + N^2(k^2 + l^2)}{k^2 + l^2 + m^2 + \frac{1}{4H^2}} = \left(f^2 + \frac{N^2(k^2 + l^2)}{m^2 + \frac{1}{4H^2}} \right) \frac{m^2 + \frac{1}{4H^2}}{k^2 + l^2 + m^2 + \frac{1}{4H^2}} \quad (28)$$

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