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Geoscientific Model Development Discussions

Interactive comment on "*HydrothermalFoam* v1.0: a 3-D hydro-thermo-transport model for natural submarine hydrothermal systems" *by* Zhikui Guo et al.

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As both reviewers had questions about the energy equation, we here provide the full derivation. Most of it is based on the derivation given in Bird (2002) but we here provide some extra information on the individual steps. Starting point is the equation of change of internal energy, which in turn results from subtracting the mechanical energy balance from the full energy balance. It is eqn. 11.2-1 in Bird's book, here written for a porous medium. All the symbols definition and unit are listed in Table 1.

$$\frac{\partial(\varepsilon\rho_f u_f + (1-\varepsilon)\rho_r u_r))}{\partial t} = -\nabla\cdot(\vec{q}) - \nabla\cdot(\rho_f u_f \vec{U}) - p\nabla\cdot\vec{U} + \frac{\mu_f}{k}\parallel\vec{U}\parallel^2 \qquad (1)$$

The left-hand side describes the change in internal energy in the porous medium, which is related to conduction (1. term on Rhs), advection (2. term), pressure-volume work (3. term), and viscous dissipation (4. term).
Step 1 is to substitute the thermodynamic identity

$$u = h - pV = h - \frac{p}{\rho_f} \qquad (2)$$

into eqn. (1):

$$\frac{\partial(\varepsilon\rho_f\left(h_f - \frac{p}{\rho_f}\right) + (1-\varepsilon)\rho_r u_r))}{\partial t} = -\nabla\cdot(\vec{q}) - \nabla\cdot(\rho_f\left(h_f - \frac{p}{\rho_f}\right)\vec{U}) - p\nabla\cdot\vec{U} + \frac{\mu_f}{k}\parallel\vec{U}\parallel^2 \qquad (3)$$

Separating the terms and assuming constant porosity yields:

$$\varepsilon\frac{\partial\rho_f h_f}{\partial t} - \varepsilon\frac{\partial p}{\partial t} + (1-\varepsilon)\frac{\partial\rho_r u_r}{\partial t} = -\nabla\cdot(\vec{q}) - \nabla\cdot(\rho_f h_f\vec{U}) + \nabla\cdot(p\vec{U}) - p\nabla\cdot\vec{U} + \frac{\mu_f}{k}\parallel\vec{U}\parallel^2 \qquad (4)$$

Little rearrangement:

$$\varepsilon\frac{\partial\rho_f h_f}{\partial t} + (1-\varepsilon)\frac{\partial\rho_r u_r}{\partial t} = -\nabla\cdot(\vec{q}) - \nabla\cdot(\rho_f h_f\vec{U}) + \varepsilon\frac{\partial p}{\partial t} + \vec{U}\cdot\nabla p + \frac{\mu_f}{k}\parallel\vec{U}\parallel^2 \qquad (5)$$

Now we open the derivatives for the $\rho_f h_f$ terms:

$$\varepsilon\rho_f\frac{\partial h_f}{\partial t} + \varepsilon h_f\frac{\partial\rho_f}{\partial t} + (1-\varepsilon)\frac{\partial\rho_r u_r}{\partial t} = -\nabla\cdot(\vec{q}) - \rho_f\vec{U}\cdot\nabla h_f - h_f\nabla\cdot(\rho_f\vec{U}) + \varepsilon\frac{\partial p}{\partial t} + \vec{U}\cdot\nabla p + \frac{\mu_f}{k}\parallel\vec{U}\parallel^2 \qquad (6)$$

and bring the time derivative of density to the rhs while using mass conservation

$$\varepsilon \frac{\partial \rho_f}{\partial t} + \nabla \cdot (\vec{U} \rho_f) = 0 \tag{7}$$

so that the terms in red drop out, which yields:

$$\varepsilon \rho_f \frac{\partial h_f}{\partial t} + (1-\varepsilon) \frac{\partial \rho_r u_r}{\partial t} = -\nabla \cdot (\vec{q}) - \rho_f \vec{U} \cdot \nabla h_f + \varepsilon \frac{\partial p}{\partial t} + \vec{U} \cdot \nabla p + \frac{\mu_f}{k} \parallel \vec{U} \parallel^2 \tag{8}$$

Equation 8 is the equation of change of internal energy written in term of specific enthalpy - except for the solid term, which we will treat later. The next step is to use the standard thermodynamic identity to move from enthalpy to specific heat (cf. eqn. 9.8-7 in Bird's book):

$$dh_f = \frac{\partial h_f}{\partial T}_p dT + \left(\frac{\partial h_f}{\partial p}\right)_T dp = c_p dT + \left[V - T\left(\frac{\partial V}{\partial T}\right)_p\right] dp \tag{9}$$

Substituting eqn. (9) into (8) results in the specific heat being outside the derivatives (cf. eqn. 11.2-4 in Bird's book):

$$\varepsilon \rho_f c_p \tfrac{\partial T}{\partial t} + \varepsilon \rho_f \left(\tfrac{\partial h_f}{\partial p}\right)_T \tfrac{\partial p}{\partial t} + (1-\varepsilon)\tfrac{\partial \rho_r u_r}{\partial t} = -\nabla \cdot (\vec{q}) - \rho_f c_p \vec{U} \cdot \nabla T - \rho_f \left(\tfrac{\partial h_f}{\partial p}\right)_T \vec{U} \cdot \nabla p$$
$$+\varepsilon \tfrac{\partial p}{\partial t} + \vec{U} \cdot \nabla p + \tfrac{\mu_f}{k} \parallel \vec{U} \parallel^2 \tag{10}$$

Some re-arrangement using eqn. (9) :

$$\varepsilon \rho_f c_p \tfrac{\partial T}{\partial t} + (1-\varepsilon)\tfrac{\partial \rho_r u_r}{\partial t} = -\nabla \cdot (\vec{q}) - \rho_f c_p \vec{U} \cdot \nabla T + \varepsilon \rho_f T \left(\tfrac{\partial \frac{1}{\rho}}{\partial T}\right)_p \tfrac{\partial p}{\partial t} + \rho_f T \left(\tfrac{\partial \frac{1}{\rho}}{\partial T}\right)_p \vec{U} \cdot \nabla p$$
$$+\tfrac{\mu_f}{k} \parallel \vec{U} \parallel^2 \tag{11}$$

Next we use a logarithmic derivative for the $\frac{\partial \frac{1}{\rho}}{\partial T}$ terms,

$$\rho_f T \left(\frac{\partial \frac{1}{\rho_f}}{\partial T}\right)_p = -\frac{T}{\rho_f}\left(\frac{\partial \rho_f}{\partial T}\right)_p = -\left(\frac{\partial ln \rho_f}{\partial ln T}\right)_p \tag{12}$$

and combine equation 11 (terms in blue and red) with equation 12 to get:

$$\varepsilon \rho_f c_p \frac{\partial T}{\partial t} + (1-\varepsilon)\frac{\partial \rho_r u_r}{\partial t} = -\nabla \cdot (\vec{q}) - \rho_f c_p \vec{U} \cdot \nabla T - \left(\frac{\partial ln \rho_f}{\partial ln T}\right)_p \left(\varepsilon \frac{\partial p}{\partial t} + \vec{U} \cdot \nabla p\right) + \frac{\mu_f}{k} \parallel \vec{U} \parallel^2 \tag{13}$$

which is (almost) the energy equation provided in the main text. The final step is to treat the solid terms. Here we again use the standard thermodynamic identity (cf. eqn. 45.2-45.7 in Feynman et. al., 2011):

$$du = c_v dT + \left[p - T\frac{\partial p}{\partial T}\right]_p dV \tag{14}$$

As the solid is assumed incompressible, the second term vanishes and $c_v = c_p$, so that

$$(\varepsilon \rho_f c_p + (1-\varepsilon)\rho_r c_{pr}) \frac{\partial T}{\partial t} = -\nabla \cdot (\vec{q}) - \rho_f c_p \vec{U} \cdot \nabla T - \left(\frac{\partial ln \rho_f}{\partial ln T}\right)_p \left(\varepsilon \frac{\partial p}{\partial t} + \vec{U} \cdot \nabla p\right) + \frac{\mu_f}{k} \parallel \vec{U} \parallel^2 \tag{15}$$

which now finally is the equation 4 given in the main text. In addition, $\frac{\partial ln \rho_f}{\partial ln T} = \frac{T}{\rho_f}\frac{\partial \rho_f}{\partial T} = -T\alpha_f$, where $\alpha_f \equiv -\frac{1}{\rho_f}\frac{\partial \rho_f}{\partial T}$ is defined as fluid thermal expansivity which is used in the equation 7 in the main text.

Table 1. Definitions and units of variables

Symbol	Definition	Unit
t	Time	s
ε	Porosity	1
$\vec{q}$	Conductive heat flux	$W\ m^{-2}$
$\vec{U}$	Velocity	$m\ s^{-1}$
p	Pressure	Pa
T	Temperature	K
	Fluid and rock properties	
ρ	Density	$kg\ m^{-3}$
h	Specific enthalpy	$J\ kg^{-1}$
u	Specific internal energy	$J\ kg^{-1}$
V	Specific volume	$m^3\ kg^{-1}$
c_p	Specific heat for a constant pressure process	$J\ kg^{-1}\ K^{-1}$
c_v	Specific heat for a constant volume process	$J\ kg^{-1}\ K^{-1}$
α	Thermal expansivity	K^{-1}
Subscript f and r denote fluid and solid(rock), respectively		

References

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