Review – Cheng, et al., A full Stokes subgrid model for simulation of grounding line migration in ice sheets using Elmer/ICE (v8.3)

The manuscript aims to present a numerical scheme to deal with the friction inside elements partly floating in a (full-)Stokes formulation for the marine ice sheet simulation. The formulation and results are carried out in a 2D vertical domain, and possible extension to 3D domain is discussed. Overall the paper is well written, and the figures are well visible. The results are compared to a related study (Glagliardini et al, 2016), where no subgrid scheme is applied: while in the latter the results are achieved using high mesh resolution (50 to 25 m), the current manuscript presents "similar" results using 4000 to 1000 m as mesh resolution.

General comments:

- The numerical scheme as well as the equations being solved are presented, although some corrections should be done to make them clearer. Also, it is not clear if there are some iterative steps between the GL position computation and the solver of the FS equations. An overview of the framework involving all the solver processes (each solver step) employed could be useful to make it clearer.
- With the results presented along the manuscript it is hard to analyze the convergence (and consistency) of the subgrid scheme proposed. Also, as mentioned above, the overall explanation on how the entire mathematical problem is solved doesn't help this analyze and could compromise the reproducibility of the results. The results seem promising, however.
- The Introduction section must be improved. The reading is impact by some zigzags. Some simplifications could be done, starting from an overview of the problem and going to the specific problem that is being solved in the manuscript.
- The citation style along the manuscript should be corrected, e.g., in line 89, Hutter (1983) => (Hutter, 1983), and elsewhere.

Specific comments:

2 Ice model

2.1 The full Stokes (FS) equations

- line 89: "2D" => "2D vertical" (and elsewhere)
- line 90: "ice Ω" => "ice domain Ω"
- line 92: the notation here is confused. It should be:

$$\sigma = \tau - p\mathrm{I}$$

where τ is the deviatoric stress tensor, given by

$$\tau = 2\eta\dot{\varepsilon}$$

where $\dot{\varepsilon}$ is the strain rate tensor, defined as

$$\dot{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}})$$

and η is the ice viscosity given by

$$\eta = \frac{1}{2}A^{-\frac{1}{n}}\,\dot{\varepsilon}_e^{\frac{1-n}{n}}$$

being $\dot{\varepsilon}_e$ defined as

$$\dot{\varepsilon}_e = \sqrt{\frac{1}{2}\mathrm{tr}(\dot{\varepsilon}\dot{\varepsilon}^{\mathrm{T}})}$$

Please, don't use "τ" as strain rate factor, even because it is used in Eq. 10 (and elsewhere) as stress.

2.2 Boundary conditions

- line 101: the boundary Γ is not represented in Figure 1 (neither Γ_s and Γ_b)
- line 101: "In a 2D case" => "In the 2D vertical case"
- line 102: "y is constant in the figure" => "the ice sheet geometry is constant in y"
- line 102 (and elsewhere): Please, use "ice surface" instead of "upper boundary" or "upper surface", and "ice base" instead of "lower boundary" or "lower surface".
- line 104: The notation here doesn't help. Please, use $\mathbf{f}_s$ only for the forcing applied at the ice surface, Γ_s. For the floating part of the ice base, Γ_{bf}, use, for example, $\mathbf{f}_{bf}$.
- line 106: it is good explaining (with few words) the meaning of σ_{nn} (the normal component of the force/stress acting on the ice base …), σ_{nt} (the parallel component of the force/stress acting on the ice base …), and u_t (the parallel component of the ice velocity at the ice base …).
- line 108: the same for u_n
- line 112: "The GL is located where" => "At the GL,"
- line 112: "In 2D" => "In 2D vertical"
- line 114: "With the ocean surface at $z = 0$, p_w … " => "The ocean surface is at $z = 0$, and p_w …"
- line 115: "gravitation constant" => "gravitational acceleration"

2.3 The free surface equations

- line 116: maybe change “free surface equation” to “ice surface and ice base equations”
- line 119 (and elsewhere): “free surface” => “ice surface”
- line 122 (and elsewhere): “lower surface” => “ice base”
- line 122: z_b is negative if below sea level, right?
- line 124: actually, there is just ablation (basal melt) at Γ_{bf}

2.4 The solution close to the grounding line

- line 128: “The 2D” => “The 2D vertical”
- line 130: “the bedrock” => “the grounded part”
- line 130: Which “simple equation” in Schoof 2011? It is not clear (maybe you meant “simple relation”)
- line 132: “By adding higher order terms” where? The phrase is not clear
- line 132: “… Archimedes’ floatation condition … is not satisfied …” where? In Schoof 2011 analysis? Or considering all the terms in FS? The phrase is not clear
- line 134: does the rapid variation of w appear in Schoof 2011 analysis? The phrase is not clear
- line 136: “height” => “thickness”
- line 136: “length” => “horizontal length”
- line 140: basically, is it assumed that the vertical normal stress (σ_{22} or σ_{zz}) is hydrostatic? Or at least at the ice base (boundary)? See Greve and Blatter, 2009, Eq. 5.59, for example.
- line 141: Does “bedrock” here mean grounded ice or the bedrock, $b(x, z)$? Also, I didn’t understand the reference to Eq. 4 and Eq. 6.
- line 143: “Introduce” => “Introducing”
- line 145: “approximate” => “approximating” and “let” => “letting”, and “… water surface. Then” => “… water surface, yields”
- line 145: “water surface” => “sea level”
- line 145: What is the reason to “approximate z_s and z_b linearly in x”? Both z_s and z_b are linear if the elements are geometrically linear (i.e., the edges of the elements remain straight). Is this the case being referred here? Or are you using this argument to simplify the expression of σ_{zz} (χ)?
- line 147: Eq. 12 is a hydrostatic condition to estimate the GL position, right? Why is H_{bw} used instead of the bedrock coordinate, $b(x)$? Using the bedrock instead of H_{bw}, the function χ should be positive for floating ice, and negative for grounded ice. The GL position is found for $\chi = 0$. Using H_{bw}, it is not clear how to find the GL position, since $\chi = 0$ for the floating ice, including the GL position, x_{GL}. (In fact, this is pointed out in line 272)

3 Discretization by FEM

3.1 The weak form of the FS equations

- line 155: who is "n"? Is "n" equal to "p" in Chen et al. 2013 notation?
- line 156: please check the definition of the spaces here. It is not clear if "k" means k-integrable functions. Also, the space Q_{k^*} and k^* are not well defined. Is it imposed divergence free for Q_{k^*}? Or is only the L^2 norm required for the pressure space?
- line 161: please, change the notation of the strain rate tensor (as already mentioned above)
- line 163: "size" => "value", and "application" => "physical problem"
- line 163: an observation could be added here, something like: "the sensitivity of the GL positions for different values of γ_0 is shown in sect. 5"

3.2 The discretized FS equations

- line 169: the space M_{k^*} is not defined
- line 170: are the vertical layers equally spaced, for a given x?
- line 174: "ice" => "ice sheet"
- line 177: (Eq. 15) the integral limits should something like $s(x_i)$ to $s(x_{GL})$, and $s(x_{GL})$ to $s(x_{i+1})$, where s is the lowest edge (the ice base) of the element $\mathcal{E}_i$ where the GL is located (between x_i and x_{i+1}). Maybe a note could be added in the text instead of changing it in the equations (in line 171, for example).
- line 178: this paragraph is not clear. What does it mean "strong formulation"? Also, the basis functions at the lowest edge of the element $\mathcal{E}_i$ (the edge that represents the ice base) are linear, and defined between $s(x_i)$ and $s(x_{i+1})$, right? (here, s is the coordinate along this element's edge). So, even if the integral is split at $s(x_{GL})$, the GL position at the element's edge, there are contribution of the integral on both nodes, $s(x_i)$ and $s(x_{i+1})$, since the basis functions are defined between $s(x_i)$ and $s(x_{i+1})$ (unless additional or modified basis functions are used, similar to X-FEM for example, which it seems it is not the case here). For example, let's take the last integral in Eq. 15, and let's assume (for simplicity) the base is perfectly horizontal between x_i and x_{i+1}. The normal vector at the base is $\mathbf{n} = [0\ \ 1]^{\mathrm{T}}$, and $\mathbf{u} \cdot \mathbf{n} = w$ (vertical velocity). Assuming linear piecewise basis functions (ϕ) at the nodes $s(x_i)$ and $s(x_{i+1})$, we have $w = w_i\phi_i + w_{i+1}\phi_{i+1}$, and the integral is $\int p_w(w_i\phi_i + w_{i+1}\phi_{i+1})ds$ at that edge, which means that there are contributions of the integral on both nodes, i and $i+1$. The same happens for the other integral (whose limits are $s(x_i)$ and $s(x_{GL})$. Could you please make that paragraph clearer?
- line 182: "non-linear" => "nonlinear"
- line 199: is "d_j" here referred to "d", the distance between the base and the bedrock?
- line 199: it is not totally clear how the complementarity problem is solved. During the Newton iterations to solve both $\mathbf{u}$ and p, the distance d is kept constant, right (and consequently p_w at the base)? Are σ_{nn} and u_n updated at every Newton iteration? So is the complementarity problem (Eqs. 16, 17 and 18) solved at each Newton iteration? How is the GL position defined during this process? By using the function χ (Eq. 11)? Could you please explain these solver steps?

- line 199: the term "grounded mask" is not used along the text. Could you please explain or change this term? Please, avoid different definitions along the text.

3.3 Discretization of the advection equations

- line 203: are you using a complete stabilization scheme, like the Streamline Upwind Petrov-Galerkin (SUPG) scheme, or a scheme based on adding an artificial diffusion, line Artificial Diffusivity or Streamline Upwind?
- line 205: are both advection equations solved together (fully coupled)? Please, could you make it clearer?
- line 209: What does it mean: "The spatial derivative of z_c is approximated by FEM"? Is the derivative of z_c that one provided by the derivative of the basis functions? Or is it applied some gradient recovery method? Please, could you explain it?
- line 215: "Insert" => "Inserting" and "to obtain" => "yields"
- line 217 and Eq. 22: Why is the notation in t different here? In Eq. 19 there is a t^{n+1}; here, t^n.
- line 218: z_{bx} is not defined in time here (implicit or explicit?)
- line 219: it is not clear, but it seems that both advection equations (or at least Eq. 22) are solved together with the FS equations, Eq. 14. Please, could you explain this? Eq. 19 is semi-implicit, i.e., Eq. 14 is solved first for velocity and pressure, but it is a bit confusing with Eq. 22.
- line 220: "Assume" => "Assuming"
- line 220: "… is small. The timestep …" => "… is small, the timestep …"
- line 223: "Divide" => "Dividing" (and elsewhere along this paragraph and below)
- line 237 (and all paragraph): which exactly scheme is used after all? The semi-implicit for the ice surface and ice base (Eq. 19), or the scheme related to Eq. 22? It is not clear. Could you please add a sequence of the numerical scheme, including the FS equations and the complementarity equations? How is the GL position calculation used here?

4 Subgrid modeling around grounding line

- line 242: "Subgrid modeling around grounding line" => "Subgrid scheme around grounding line"
- line 243: what GL parameterization in Seroussi et al., 2014? SEP1, SEP2 or SEP3?
- line 246: please, delete "exact"
- line 245 to 247: "In the Stokes equations, the hydrostatic assumption may not be valid, so the exact GL position can not be determined by simply checking the total thickness of the ice H against the depth below sea level $H_{bw} = -z_b$." But this assumption is used to deduce Eq. 12.
- line 249: the indicator χ defined here is different to the indicator defined by Eq. 11 or 12. Also, if $\tau_{22} - p$ is defined by Eq. 10, we have a hydrostatic assumption for σ_{22} (=σ_{zz}, right?). Then, at the end, is a hydrostatic assumption used to define the GL position?
- line 250: "since the slope of the bedrock is small" is it the argument to justify the hydrostatic approximation at the ice base? Could you please explain this phrase?
- line 251: "lower surface" => "ice base"

- line 251: "$z_b > b$" is not the only condition to define the boundary conditions, right? Because, even in the situation of Figure 2 (upper panel), the net force at node x_{i+1} could be pointing outward, i.e., forcing the node to be grounded (e.g., an advance phase of the ice sheet).
- line 253: is the "net force" represented by the "arrows" in Figures 2 and 3?
- line 257: "contact with the bedrock" means $z_b = b$, right?
- line 259: "GL element" is the same element $\mathcal{E}_i$ defined in the line 173, right? Please, try to simplify the number of definitions along the text.
- line 261: The "true position" term here is complicated. Actually, in any fixed mesh with any finite resolution, the "true position" of the GL is not defined. Also, this paragraph seems redundant and could be deleted.
- line 264: Again, the definition of χ. Also, Eq. 11 is a hydrostatic assumption of χ.
- line 266: Note that $\chi(x_i) > 0$ in this case because it is a discrete case (x_{GL} is not perfectly defined). Using Eq. 12, in a continuous case (and with a perfect definition of x_{GL}), $\chi \leq 0$, as written in line 148. Maybe a note could be added to avoid confusion.
- line 268: "floating boundary condition" => "floating condition, Eq. (11)" (maybe)
- line 289: The correction made in χ is only in the pressure water, right? σ_{nn} continues as hydrostatic, Eq. 11 or 12, right?
- line 272: why is the bedrock elevation not used in every case (i and ii), if this is the most generic way to solve for x_{GL}?
- line 273: "linear functions" => "linearized functions"
- line 274: "As the GL always rests on the bedrock" in theory or ideally, right? Then $p_b(x_{GL})$ is equal to $p_w(x_{GL})$ only when $b(x_{GL}) = z_b(x_{GL})$, what is not the case in the (most) discretization representations. Maybe what you are saying here is that in the solution of x_{GL} using the linear interpolation of χ, $z_b(x)$ should be equal to $b(x)$ at x_{GL}, what is not true due to the linear representation of $z_b(x)$, mainly in a coarse mesh resolution (even in a fine resolution, a residual will exist). I think this phrase or even this paragraph could be rewritten. Also, σ_{nn} here is that one computed by Eq 10, right? So σ_{nn} is hydrostatic, right?
- line 278: what correction do you refer here? Please, could you make this paragraph clearer?
- line 281: Then you increase the number of integration (Gauss) points, right? Similar to SEP3 scheme in Seroussi et al., 2014. What order do you use? Is there any sensitivity in GL positions for different orders?
- line 282: I understand the motivation of "smoothing" the friction coefficient β at the GL region, mainly when a Weertman-type friction law is employed (even because β "should" be zero at the first floating node, so it seems that 1/2 comes from a linear interpolation of β between the last grounded and the first floating nodes). But this "smoothing" effect should already be "captured" by the subgrid scheme you are using (i.e., the basal friction on the GL element is "weighted" by the integration points; it is not a "smoothing" effect in fact, but a reduction of the friction inside the GL element). I am not sure about the effect of multiply β by 1/2. Imagine the case where the GL is very close to the first floating node, x_{i+1}, and in the next time step, the GL moves to the next (floating) element (defined between x_{i+1} and x_{i+2}). So, there is a "jump" in the basal friction

on the previous GL element ($\in [x_i \; x_{i+1}]$), from an almost fully grounded case (with friction coefficient β multiplied by 1/2) to a fully grounded case (friction coefficient equal to β). Have you tested without this "smoothing" (multiplication)?
- line 284: this paragraph is not totally clear. It is not clear how the integral of Eq. 15 is discretized/applied on the GL element. Could you please make this clearer using the same notation of Eq. 15?
- line 290: this paragraph could be a summary of how the equations are solved, but it not totally clear. Are the advection equations solved together with the FS equations, or in a semi-implicit manner? Does the "fixed-point" here refer to the Newton method (line 197) or to a Piccard-like scheme? Also, the high integration order is applied only in the GL element, right?

5 Results
- line 296: "and comparison" => "and a comparison" (maybe)
- line 299: "20 vertical extruded layers" equally spaced?
- line 301: I liked this analysis on γ_0. I suspect the value of γ_0 should be updated according to the order of magnitude of the matrix coefficient about which the Nitche's term is added (in the stiffness matrix); since it is nonlinear, the order of magnitude of the (stiffness) matrix coefficients change during the simulation (maybe even in each nonlinear iterations). But this is a study to be carried out in the future.
- line 306: "mesh sizes" => "mesh resolutions". The same for line 309.
- line 308: "purple and pink" => "purple and pink, respectively"
- line 308: "We achieve similar GL migration results … with at least 20 times larger mesh sizes." It is an impressive result, and it seems very promising! Some questions arise here. A) It is not expected monotonicity in terms of GL convergence, but it is hard to analyze the convergence looking only at Figure 5. What about plotting also a figure similar to Figure 2 of Gagliardini et al. 2016? B) The GL positions in both phases (advance and retreat) are quite similar using mesh resolution equal to 2 km, but the same is not observed when the resolution is increased (1 km), although the GL positions seem to move (converge?) to the central value (~730 km) with mesh resolution. Is the bedrock description ($b(x)$, given by Eq 16 in Pattyn et al. 2012, right?) at the same resolution of the mesh? If yes, increasing the mesh resolution also increases the bedrock resolution, and we should expect GL convergence to the interval obtained by Gagliardini et al. 2016 (considering that they also increased the bedrock resolution). I think a way to "strengthen" the results is to run the same experiments with one more level of refinement, i.e., with mesh resolution equal to 500 m (if possible, even 250 m). If the bedrock is kept at the coarse resolution (i.e., 4 km in your case here), then the results would not necessarily converge to the Gagliardini interval, and the comparison with their results doesn't seem adequate (although the convergence with mesh resolution should be easier analyzed since the geometry/bedrock is the same for all meshes). Another way to verify the convergence and consistency of the numerical scheme is to run the same kind of experiments using a single-slope bedrock (like MISMIP3D, for example), with different mesh resolutions.

- line 314 (and Fig. 6): it seems that x_{GL} is estimated using $\rho H = \rho_w \, H_{bw}$. But, it seems that this is not observed in Fig. 6 (right panel) for the GL position presented. Why? In Fig. 6, I expected the red-dashed line (GL) to cross both the purple-dash-dotted line and the green line. Or maybe I am missing something here …
- line 320: “top and bottom” => “surface and base” (and elsewhere in this paragraph)
- line 321: “The horizontal velocities on the two surfaces are similar with negligibly small differences on the floating ice.” as expected for the floating ice, right?
- line 323: “… the rapid variation is resolved by the 1 km mesh size.” I don’t think “resolved” is the right term here. The discontinuity at x_{GL} is not resolved with a continuous space, and the variation of w in x doesn’t seem to be a polynomial-type. Maybe “enough approximated” is the term you meant (although “enough” is a matter of discussion, indeed).

6 Discussion

- line 326: “floatation criterion” => “hydrostatic floatation criterion”
- line 326: “Depending on how many of the nodes that are floating, the amount of friction in the triangle is determined.” Maybe this could be rewritten. Basically, the amount of friction is computed according to the grounded are in partly floating elements. And Seroussi et al., 2014, used different techniques, based on the FEM, to compute this amount of friction.
- line 327: “Also, a higher order polynomial integration over the triangles in FEM allows an inner structure in the triangular element.” I didn’t understand this phrase; what does it mean “inner structure” here?
- line 330: “… If $\chi \geq 0$ …” χ is higher than 0 if the water pressure is computed using the bedrock elevation ($b(x)$, as described in line 269). Otherwise, $\chi = 0$ on the floating nodes (as in line 148, unless in the case as shown in Fig. 6).
- line 337: “model inaccuracy” => “numerical error of the model”
- line 340: The subgrid also helps the convergence in comparison to non-subgrid scheme. See for example the comparison between NSEP and SEP1 or SEP2 in Seroussi et al., 2014 (e.g., Fig 2)
- line 340: How does your subgrid implementation compare to the “DI implementation” tested in Gagliardini et al. 2016?

7 Conclusions

- line 341: “Subgrid models at the GL” => “Subgrid schemes to model the GL dynamics”
- line 341: “3D flow” actually it is also a SSA-2D flow
- line 342: “3D flow” same here; BISICLES is a “2 1/2 flow”, but the grid is 2D (plan x-y)
- line 345: “subgrid model in 2D” => “subgrid scheme in 2D vertical”
- line 346: Note that $\tilde{\chi}$ here is the modified version of χ. Please, try to condensate and simplify the definitions along the text, and try to use just one
- line 348: “method” => “numerical scheme”
- line 348: delete “in 2D”
- line 348: “The data” => “The model setups”

- line 350: “with subgrid modeling” => “using the subgrid scheme”
- line 351: please, delete “Without further knowledge of the basal conditions and detailed models at the GL”.
- line 352: “… approximation of the GL position.” => “…approximation of the GL position, and accelerates the GL position convergence in comparison to schemes where the GL relies just on element nodes.” **Note that this last phrase (suggestion) only makes sense if more numerical tests are performed, helping the convergence analysis of the proposed subgrid scheme.**

As a last suggestion, maybe change “subgrid model” in the title to “subgrid scheme”

References (used here)

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