

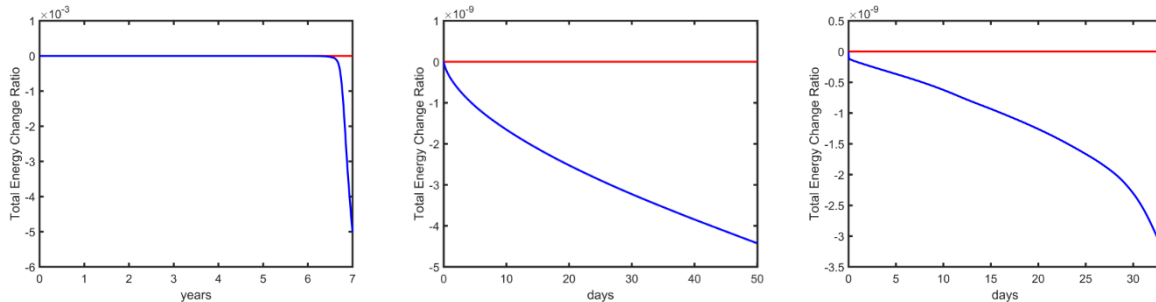
Thank you very much for reading our manuscript meticulously, those problems you found and the advices you mentioned, help us a lot to improve the expression and strictness to the manuscript. There are some opinions we'd like to discuss and share with you.

The main theme of this manuscript is finding out a method to extend the square conservation scheme, from regular latitude-longitude grid to an arbitrarily structured C-grids dynamic core, TRiSK, meanwhile, the intrinsic property of TRiSK (including accuracy of operators and conservation properties) should not be broken down.

There are some problems with the presentation in the manuscript (details below), which it should be possible to fix. However, the key ideas needed to obtain exact energy conservation on arbitrary meshes have been around for a while; this paper merely brings them together. Also, temporal truncation errors tend to be much smaller than spatial truncation errors in atmospheric models, so only a small improvement (if any) is obtained by replacing an energy-conserving spatial discretization by an energy-conserving space-time discretization, (as the results in this paper confirm). Thus, I think the manuscript lacks the originality and significance needed to justify publication in GMD.

Reply:

The most of so-called energy conservation schemes on arbitrary meshes merely conserve total energy within time truncation error, i.e. (Ringler et al,2010), (Thi-Thao-Phuong Hoang,2019), the energy is still slightly dissipative during temporal integration. As Eq.(24) in the manuscript, the energy is strictly conserved only if  $2\tau_n(\varphi^n, F^n) + \tau_n^2 \|\varphi^n\|^2 = 0$ , unfortunately, most of temporal integration schemes do not satisfy this condition. Conserving energy within time truncation error is not equivalent to strict/exact energy conservation, the former allows slightly energy dissipative/anti-dissipative during temporal integration, the later conserving total energy in round-off error. In the manuscript, Figure 3c, Figure 4c and Figure 5c (the same as the following figures, from left to right) are showing the differences between “Conserving energy within truncation error” (blue line) and “Strictly conserving energy” (red line).



Indeed, there are some methods to exactly conserve energy on arbitrary meshes, i.e. taking energy as an evolution variable, based on conservation law, the energy flux is balanced between each cell, therefore energy is conserved anywhere and anytime (Satoh, 2004, Section 1.2.3). But these methods are not quadratic conservation in mathematics. In the shallow water system,

one can obtain the exact energy conservation by replacing the continuous equation by energy equation, but this method sacrifices mass conservation; or in another way, replacing the momentum equation by energy equation, but the flow direction will not be able to be determined, and sometimes worse situation appears, since the lack of constrain from momentum equation, potential energy could be greater than total energy, which result in the wind speed becomes imaginary number.

- 5 By implementing the square conservation scheme, neither momentum equation nor continuous equation needs to be replaced by energy equation, the total energy is strictly conserved, rather than conserved within time truncation error, meanwhile, there are not influences to the other conservative properties, such as mass and absolute vorticity.

About the originality. The prerequisite of implementing square conservation scheme is that the spatial discrete operator must be anti-symmetrical, but it is hard to construct an anti-symmetrical operator on quasi-uniform grids directly, therefore we try to find another way to obtain the anti-symmetrical operator. Energy conservation is one of the intrinsic properties of TRiSK shallow water dynamic core, and as we mentioned in Section 3.2, Eq.(13) is the key to connecting square conservation and energy conservation, by using this simple combination of original TRiSK spatial discrete operators, the anti-symmetrical operator is built.

- 10 Note that, for constructing the anti-symmetrical operator in shallow water system, the units of the evolution variables must be unified, otherwise, the addition is not able to operate between different variables, this is the reason we take IAP transformation. Improving the conservation property is not like improving accuracy of the model, the convergence rate of spatial discrete operator, and the accuracy of temporal integration scheme are not changed in our study. Indeed, the reductions of errors are not such significant, but the physical characteristics are more analogous to the real system. The differences are not obvious in short term simulation, but in long term simulation, the advantage of strict energy conservation scheme may be huge, this is intuitively showed by the numerical test in (Wang,1996), which we'd like to share with you in Response for specific comment #2.

In the following content, we response your specific comments.

Response for specific comment #1:

- 25 Sections 3.1, 3.2. The notation  $\mathcal{L}F$ ,  $\mathcal{M}u$ ,  $\mathcal{N}\phi$  suggests that  $\mathcal{L}$ ,  $\mathcal{M}$ ,  $\mathcal{N}$  are linear functions of  $F$ ,  $u$ , and  $\phi$  respectively. In fact they are all nonlinear functions. Moreover,  $\mathcal{M}$  and  $\mathcal{N}$  are actually functions of both  $u$  and  $\phi$ . These two sections are over-elaborate and presented in a very confusing way. In several places it is not obvious what is assumed and what is claimed to be proved. All that is really needed is the fact that the energy at a point can be written as a squared quantity by making a certain change of variables.

- 30 Reply:

Thank you for reminding, indeed, the derivation does not depend on the linear operation, but indeed the expression is not strict enough. The following expression is better

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{M}(\phi, u) = 0 \\ \frac{\partial \phi}{\partial t} + \mathcal{N}(\phi, u) = 0 \end{cases},$$

For simplify expression, we write  $\mathcal{M} = \mathcal{M}(\phi, u)$ ,  $\mathcal{N} = \mathcal{N}(\phi, u)$

$$\frac{\partial U}{\partial t} = \sqrt{\phi} \frac{\partial u}{\partial t} + \frac{u}{2\sqrt{\phi}} \frac{\partial \phi}{\partial t} = -\sqrt{\phi} \mathcal{M} - \frac{u}{2\sqrt{\phi}} \mathcal{N}$$

$$(\mathcal{L}(\mathbf{F}), \mathbf{F}) = -\left(\frac{\partial U}{\partial t}, \mathbf{U}\right) - \left(\frac{\partial \phi}{\partial t}, \phi\right)$$

$$\begin{aligned} 5 \quad &= \oint_{\Omega} -\mathbf{U} \frac{\partial U}{\partial t} - \phi \frac{\partial \phi}{\partial t} ds \\ &= \oint_{\Omega} -\mathbf{U} \left( -\sqrt{\phi} \mathcal{M} - \frac{u}{2\sqrt{\phi}} \mathcal{N} \right) + \phi \mathcal{N} ds \\ &= \oint_{\Omega} \phi u \cdot \mathcal{M} + \frac{|u|^2}{2} \mathcal{N} + \phi \mathcal{N} ds \\ &= (\mathcal{M}, \phi u) + (\mathcal{N}, E) \\ &= 0 \end{aligned}$$

10 All of the similar expressions are fixed in the new version manuscript.

About “the energy at a point can be written as a squared quantity by making a certain change of variables”, this is what we are talking about in Section 3.1, the square conservation is equivalent to energy conservation in a continuous system.

Response for specific comment #2:

15 P1 line 27, also P2 lines 3-5. The opening sentence is too categorical. For a quantity like energy or potential enstrophy, in a numerical model the total is made up of resolved and unresolved contributions. Therefore, it is not obvious that conserving the resolved contribution is necessary for a good solution; indeed, it may be necessary to dissipate the resolved contribution (e.g. to prevent ‘spectral blocking’). One can argue for a conservative numerical method by saying that we want to parameterize any dissipation, not leave it to numerical errors, but the opposite argument can also be made. Such ideas are extensively  
20 discussed in the literature.

Reply:

Indeed, there are resolvable and unresolvable energy contributions as mentioned in (Chapter 11, Lauritzen, 2010). Since the model resolution is not able to reach the molecule level, the numerical model cannot resolve all of the mass, technically, there are resolvable and unresolvable mass contributions either, as widely known, it’s hard to obtain a good result without total mass  
25 conservation. For total energy, the influence is not significant in short-term simulation, but the long-term simulation, without total energy conservation, often lead to a terrible result.

Energy conservation is one of the intrinsic conservation properties of the spatial discrete operator in TRiSK shallow water dynamic core (Section 3.7, Ringler et al, 2010), however, this property is lost during temporal integration by using original Runge-Kutta scheme. The temporal integration scheme brings time-truncation error into the model, rather than spatial discrete

operator, which means that the temporal integration scheme makes the model loses one of the intrinsic properties which is provided by spatial discrete scheme.

Figure 3c in the manuscript, a detail is that the square conservation scheme strictly conserves energy, even though the steady geostrophic flow collapses, but the original TRiSK scheme cannot maintain the total energy after collapse, this is an obvious difference between the “Conserving energy within time-truncation error” and “Strictly conserving energy”. The reason is that the square conservation scheme maintains the conservation properties of spatial discrete operators faithfully, but original temporal integration scheme does not.

An interesting example can be found in (Wang, 1996), the numerical test of the linear ODE

$$\begin{cases} \frac{dx}{dt} = -ay \\ \frac{dy}{dt} = bx \end{cases}$$

- 10 the true solution of the equation is an ellipse conform to  $bx^2 + ay^2 = c$  ( $c$  is a constant), after long term numerical simulation (after  $10^8$  steps) with original Runge-Kutta, the generalized energy tends to zero, and the solution tends to a single point (Fig. 2(a), Wang, 1996, as showing as follow), but the Runge-Kutta with square conservative property is able to maintain the generalized energy strictly conserved, and the solution is still a ellipse as initial time (Fig. 2(b), Wang, 1996).

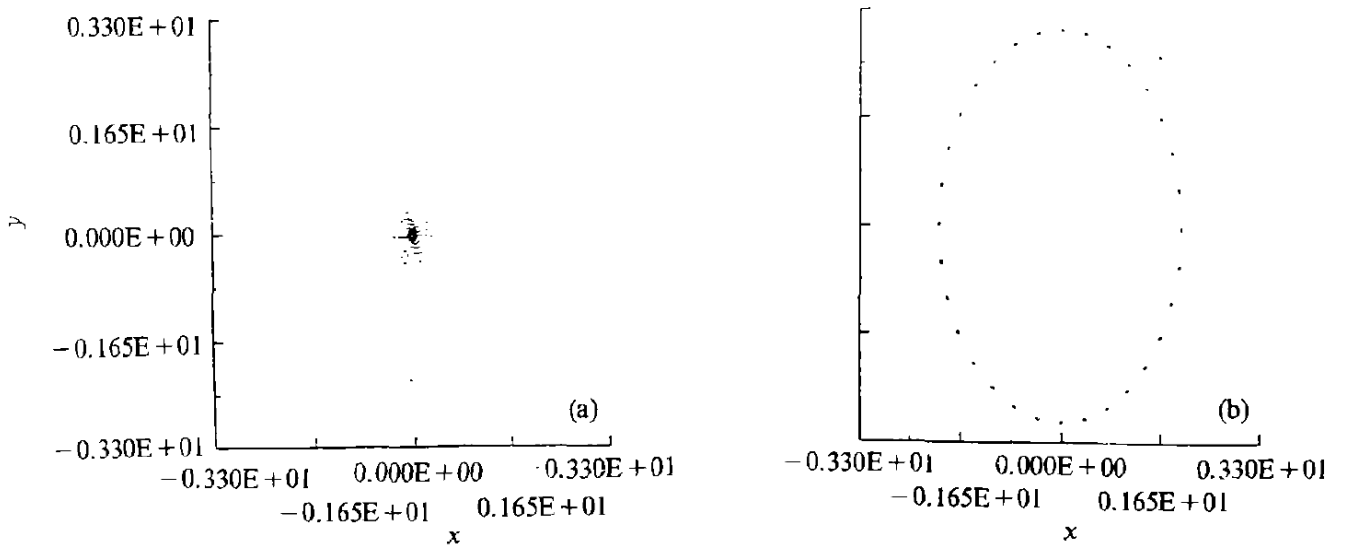


Fig. 2. Results from the  $10^8$ -step integration (printing a result per  $10^5$  steps). (a) By the old Runge-Kutta scheme, (b) by the new Runge-Kutta scheme.

- 15 This class of Runge-Kutta scheme with square conservative property is exactly what we mentioned in the manuscript Section 3.3.

In fact, there are two different forms to construct the vorticity flux term  $Q_e^\perp$  in (Section 3.9, Ringler et. al,2010), one is energy conserving form, another one is potential enstrophy conserving form, the former leads to better result than the later in all of the test cases (Fig.7, Fig.8, Fig.10, Ringler et. al, 2010), in some ways, this result proved the importance of energy conservation. In our study, the parameterize is not discussed, we consider the shallow water equations without any source/sink term of energy or mass (following Eq.(3)-(4), Ringler et. al, 2010), so maintaining the total energy conservation may be a good way to construct the dynamic core (Paragraph 4, Section1, Ringler et. al, 2010).

Response for specific comment #3:

P1 line 29, Figs. 3b, 5b, 7b, P19 line 17. On a spherical domain the vanishing of the global integral of vorticity is a purely kinematic identity. Provided the vorticity and its integral are calculated in a self-consistent way, the same result must hold in the discrete case (e.g. P10 line 20). Thus, conservation of the global integral of vorticity is a test only of the fact that the vorticity is calculated consistently; it says nothing about the properties of the numerical methods used to solve the equations.

Reply:

Total absolute vorticity conservation is one of the intrinsic properties of TRiSK shallow water dynamic core, in the manuscript, we are not trying to discuss the importance of absolute vorticity conservation, but to maintain the total energy conservation without breaking down the intrinsic properties of TRiSK, The figures and the demonstrations about the conservation of total absolute vorticity are here to prove that the square conservation scheme has no influence to the other conservation properties of TRiSK shallow water dynamic core.

Response for specific comment #4:

P1 line 29. ‘...five basic physical conserved properties’. Actually potential enstrophy is just one member of an infinite family (so-called Casimirs).

Reply:

Thank you for reminding, indeed, potential enstrophy is one member of an infinite family, in the manuscript, we are not tending to discuss all of the invariants, the description about five basic physical conservative properties is based on (Wang, 2008).

Response for specific comment #5:

P3 line 23. This form of the equations is usually called ‘vector-invariant’ (as on P19 line 16).

Reply:

Indeed, thank you very much, it has been fixed in the new version of manuscript

Response for specific comment #6:

P4. Equation (3) is inconsistent with the definition of the 2-norm on line 5. This paragraph seems to be mixing up point values and global integrals.

Reply:

Indeed, the total energy should be defined as follow

$$\oint_{\Omega} \epsilon ds = \oint_{\Omega} g \epsilon_{R10} ds = \oint_{\Omega} \phi K + \frac{1}{2} \phi^2 + \phi \phi_s ds = \|\phi K\| + \left\| \frac{1}{2} \phi^2 \right\| + \|\phi \phi_s\|$$

5 Response for specific comment #7:

Figure 1. Note that the grid used need not be uniform and regular (as suggested by the figure).

Reply:

Indeed, uniform and non-uniform grid do not influence the location of the variables and the structure of spatial discrete operators are the same as well. The square conservation scheme is available on arbitrarily structured C-grids as the title of the manuscript. As shown in Fig.1, Fig.2, Fig.3, (Ringler et. al, 2010), the regular is clear to introduce the structure of the SCVT mesh.

Response for specific comment #8:

P5. Note that the sign convection for  $u_e$  is related to the direction of the unit normal -this is crucial to get everything to work out. Also crucial for energy conservation is that  $Q_e^\perp$  is constructed to satisfy the equation on P20 line 9.

Reply:

The description of indicator function  $n_{e,i}$  for identifying the direction of  $u_e$  can be found in the end of Section 2,  $n_{e,i}$  appears in all of the correlative derivations in the manuscript.

The spatial discrete operators, that we described in the manuscript, are the same as those in (Ringler et al., 2010), we construct the anti-symmetrical spatial discrete operator by using the original spatial discrete operator in TRiSK shallow water dynamic core, therefore, all of the properties, mentioned by (Ringler et. al,2010), are still applicable in the manuscript. Indeed, there are two methods of calculating  $Q_e^\perp$  in (Ringler et al,2010), in our manuscript, the algorithm of calculating  $Q_e^\perp$  satisfies the condition to keep energy conservation, which is described in Section 3.7.2, Ringler et al. (2010).

25 Response for specific comment #9:

P5 line 22. ‘...new type of Runge-Kutta’. Not so new (1996).

Reply:

It is not so new, we try not to modify the title of (Wang, 1996), “A Class of New Explicit Runge-Kutta Schemes”, in the new version of manuscript, this expression is changed.

Response for specific comment #10:

P6 line 11. It would be helpful to give a reference for ‘IAP transformation’.

Reply:

The earliest description about IAP transformation can be found in (Section 2, Zeng, 1987), and also cited by (Wang, 2004).

Response for specific comment #11:

P11 lines 22-23. The text does not make sense - it seems to be mixing up point values and global integrals.

5 Reply:

Indeed, there is a mistake, the error measure function should be  $I(X^n) = \frac{S(X_i^n) - S(X_i^0)}{S(X_i^0)}$ , where  $S(X) = \frac{\sum_{i=1}^N X(i)A(i)}{\sum_{i=1}^N A(i)}$ ,  $X_i^n$  is the variable at the  $n$ th time point on the  $i$ th cell and  $X_i^0$  is the variable at the initial time, and  $A(i)$  is the area of the  $i$ th cell. This is similar to (135)-(140), Williamson, 1992, but the coordinate is no longer latitude-longitude, in the quasi-uniformed grid, the weight is now area of each cell. For a simpler expression  $I(X^n) = \frac{\sum_{i=1}^N (X_i^n - X_i^0)A(i)}{\sum_{i=1}^N X_i^0 A(i)}$ , the result is equivalent to the former  
10 expression.

Response for specific comment #12:

Section 5.2. The fact that the solid body rotation flow eventually breaks down, despite the conservation properties of the scheme, is intriguing. Could this be a manifestation of the ‘Hollingsworth instability’ (as discussed, for example, in Skamarock  
15 et al 2012)?

Reply:

We are not sure the connection between the collapse of steady geostrophic flow and ‘Hollingsworth instability’, but we found another way to delay the collapse, we observed that once we site the cell centers on two poles, the poles are just like the sources of errors, so we tried to rotate the mesh, and did not let any cell center site on poles, the errors was much more smaller, and  
20 the collapse has delayed obviously. Therefore, in our opinion, the principal cause of collapse is not the conservation properties, but maybe something like polar singularity, as we mentioned in the manuscript, maintaining the strict energy conservation just delays the collapse, this phenomenon needs further study.

Response for specific comment #13:

25 P19 line 26. Sign error? Line 30. What is  $h_e$ ?

Reply:

Thanks a lot.

P19 line 26 should be  $\left(U, \frac{\partial U}{\partial t}\right)_e + \left(\phi, \frac{\partial \phi}{\partial t}\right)_i = 0$ , and line 30 should be  $\left(U, \frac{\partial U}{\partial t}\right)_e = \sum_{e=1}^{nEdges} U_e \left(C_e \frac{\partial u_e}{\partial t} + \frac{u_e}{2C_e} \frac{\partial \phi_e}{\partial t}\right) A_e$

Even though, there is no influence to the result, these errors shouldn’t be happened.

30

Response for specific comment #14:

References are not in alphabetical order.

Reply:

This problem is fixed in new version of manuscript. Thank you for reminding.

#### References

- 5 Akio Arakawa and Vivian R. Lam: Computational Design of the Basic Dynamical Processes of the UCLA General Circulation Model. *Methods in Computational Physics*, 17, 173–265, 1977.  
Ringler, T. D., Thuburn, J., Klemp, J. B., and Skamarock, W. C.: A unified approach to energy conservation and potential vorticity dynamics for arbitrarily-structured C-grids, *Journal of Computational Physics*, 229, 3065-3090, <https://doi.org/10.1016/j.jcp.2009.12.007>, 2010.
- 10 T.-T.-P. Hoang et al., Conservative explicit local time-stepping schemes for the shallow water equations, *Journal of Computational Physics*, <https://doi.org/10.1016/j.jcp.2019.01.006>, 2019  
P. H. Lauritzen, et. al: *Numerical Techniques for Global Atmospheric Models*, 1st ed., Springer-Verlag Berlin Heidelberg, pp. 345-355, 2010.  
Wang, B.: Design of a new dynamical core for global atmospheric models based on some efficient numerical methods, *Science in China Series A*, 47, 4, 2004.
- 15 Wang, B., Ji, Z., and Zeng, Q.: A Class of New Explicit Runge-Kutta Schemes, *Progress In Nature Science*, 6, 195-205, 1996.  
Zeng Q., Zhang X.: Available Energy Conserving Schemes for Spherical Baroclinic Primitive Equations (In Chinese). *Scientia Atmospherica Sinica.*, 11, 113-127, 1987.
- 20 All of the references are packed in the zip file.