Supplement of

Modeling canopy-induced turbulence in the Earth system: a unified parameterization of turbulent exchange within plant canopies and the roughness sublayer

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S1 Numerical solution of Eqs. (8) and (9)

Richtmyer and Morton (1967, pp. 275–278) provide a numerical solution for Eqs. (8) and (9), common to that used for tridiagonal equations. These equations are

$$a_{1,i}\theta_{i-1}^{n+1} + b_{11,i}\theta_i^{n+1} + b_{12,i}q_i^{n+1} + c_{1,i}\theta_{i+1}^{n+1} = d_{1,i} \quad \text{(S1)}$$

$$a_{2,i}q_{i-1}^{n+1} + b_{21,i}\theta_i^{n+1} + b_{22,i}q_i^{n+1} + c_{2,i}q_{i+1}^{n+1} = d_{2,i} \quad \text{(S2)}$$

The solution involves rewriting these in the form

$$\theta_i^{n+1} = f_{1,i} - e_{11,i}\theta_{i+1}^{n+1} - e_{12,i}q_{i+1}^{n+1} \quad \text{(S3)}$$

$$q_i^{n+1} = f_{2,i} - e_{21,i}\theta_{i+1}^{n+1} - e_{22,i}q_{i+1}^{n+1} \quad \text{(S4)}$$

Here, e is a 2×2 matrix at each level i, and f is a 2×1 matrix at each level. These are found by substituting

$$\theta_{i-1}^{n+1} = f_{1,i-1} - e_{11,i-1}\theta_i^{n+1} - e_{12,i-1}q_i^{n+1} \quad \text{(S5)}$$

$$q_{i-1}^{n+1} = f_{2,i-1} - e_{21,i-1}\theta_i^{n+1} - e_{22,i-1}q_i^{n+1} \quad \text{(S6)}$$

into Eqs. (S1) and (S2) to eliminate θ_{i-1}^{n+1} and q_{i-1}^{n+1}, and then substituting the resulting equation for θ_i^{n+1} into that for q_i^{n+1} and vice versa. This gives

$$
\begin{aligned}
e_{11,i} &= c_{1,i}\left(b_{22,i} - a_{2,i}e_{22,i-1}\right) / \det \\
e_{12,i} &= -c_{2,i}\left(b_{12,i} - a_{1,i}e_{12,i-1}\right) / \det \\
e_{21,i} &= -c_{1,i}\left(b_{21,i} - a_{2,i}e_{21,i-1}\right) / \det \\
e_{22,i} &= c_{2,i}\left(b_{11,i} - a_{1,i}e_{11,i-1}\right) / \det
\end{aligned}
\tag{S7}
$$

and

$$
\begin{aligned}
f_{1,i} &= \frac{\left(b_{22,i} - a_{2,i}e_{22,i-1}\right)\left(d_{1,i} - a_{1,i}f_{1,i-1}\right) - \left(b_{12,i} - a_{1,i}e_{12,i-1}\right)\left(d_{2,i} - a_{2,i}f_{2,i-1}\right)}{\det} \\
f_{2,i} &= \frac{-\left(b_{21,i} - a_{2,i}e_{21,i-1}\right)\left(d_{1,i} - a_{1,i}f_{1,i-1}\right) + \left(b_{11,i} - a_{1,i}e_{11,i-1}\right)\left(d_{2,i} - a_{2,i}f_{2,i-1}\right)}{\det}
\end{aligned}
\tag{S8}
$$

with

$$
\det = \left(b_{11,i} - a_{1,i}e_{11,i-1}\right)\left(b_{22,i} - a_{2,i}e_{22,i-1}\right) - \left(b_{12,i} - a_{1,i}e_{12,i-1}\right)\left(b_{21,i} - a_{2,i}e_{21,i-1}\right) \tag{S9}
$$

The $\mathbf{e}$ and $\mathbf{f}$ matrices are found sequentially upward through the canopy from $i = 1$ to N with $e_{11,0} = e_{12,0} = e_{21,0} = e_{22,0} = 0$ and $f_{1,0} = f_{2,0} = 0$. Then, θ_i^{n+1} and q_i^{n+1} are calculated downward through the canopy from $i = N - 1$ to 1 using Eqs. (S3) and (S4) with $\theta_N^{n+1} = f_{1,N}$ and $q_N^{n+1} = f_{2,N}$.

S2 Algebraic derivation of Eqs. (8) and (9)

In the equations that follow, $g_{H,i}^{sun} = 2g_{b,i}\Delta L_{sun,i}$ and $g_{H,i}^{sha} = 2g_{b,i}\Delta L_{sha,i}$ are sunlit and shaded leaf conductances for sensible heat scaled to the canopy. $g_{E,i}^{sun} = g_{\ell sun,i}\Delta L_{sun,i}$ and $g_{E,i}^{sha} = g_{\ell sha,i}\Delta L_{sha,i}$ are similar conductances for evapotranspiration. The coefficients in Eqs. (8) and (9) are

$$
a_{1,i} = -g_{a,i-1} \tag{S10}
$$

$$
b_{11,i} = \frac{\rho_m \Delta z_i}{\Delta t} + g_{a,i-1} + g_{a,i} + g_{H,i}^{sun}\left(1 - \alpha_i^{sun}\right) + g_{H,i}^{sha}\left(1 - \alpha_i^{sha}\right) \tag{S11}
$$

$$
b_{12,i} = -g_{H,i}^{sun}\beta_i^{sun} - g_{H,i}^{sha}\beta_i^{sha} \tag{S12}
$$

$$c_{1,i} = -g_{a,i} \tag{S13}$$

$$d_{1,i} = \frac{\rho_m \Delta z_i}{\Delta t} \theta_i^n + g_{H,i}^{sun} \delta_i^{sun} + g_{H,i}^{sha} \delta_i^{sha} \tag{S14}$$

for temperature, and

$$a_{2,i} = -g_{a,i-1} \tag{S15}$$

$$b_{21,i} = -g_{E,i}^{sun} s_i^{sun} \alpha_i^{sun} - g_{E,i}^{sha} s_i^{sha} \alpha_i^{sha} \tag{S16}$$

$$b_{22,i} = \frac{\rho_m \Delta z_i}{\Delta t} + g_{a,i-1} + g_{a,i} + g_{E,i}^{sun} \left(1 - s_i^{sun} \beta_i^{sun}\right) + g_{E,i}^{sha} \left(1 - s_i^{sha} \beta_i^{sha}\right) \tag{S17}$$

$$c_{2,i} = -g_{a,i} \tag{S18}$$

$$d_{2,i} = \frac{\rho_m \Delta z_i}{\Delta t} q_i^n + g_{E,i}^{sun} \left[q_{sat}\left(T_{\ell sun,i}^n\right) + s_i^{sun}\left(\delta_i^{sun} - T_{\ell sun,i}^n\right)\right] + g_{E,i}^{sha} \left[q_{sat}\left(T_{\ell sha,i}^n\right) + s_i^{sha}\left(\delta_i^{sha} - T_{\ell sha,i}^n\right)\right] \tag{S19}$$

for water vapor.

Special boundary conditions are needed at the top layer ($i = N$), where $\theta_{i+1}^{n+1} = \theta_{ref}^{n+1}$ and $q_{i+1}^{n+1} = q_{ref}^{n+1}$ so that

$$c_{1,i} = 0 \tag{S20}$$

$$d_{1,i} = \frac{\rho_m \Delta z_i}{\Delta t} \theta_i^n + g_{H,i}^{sun} \delta_i^{sun} + g_{H,i}^{sha} \delta_i^{sha} + g_{a,i} \theta_{ref}^{n+1} \tag{S21}$$

$$c_{2,i} = 0 \tag{S22}$$

$$\begin{aligned} d_{2,i} = {} & \frac{\rho_m \Delta z_i}{\Delta t} q_i^n + g_{E,i}^{sun} \left[q_{sat}\left(T_{\ell sun,i}^n\right) + s_i^{sun}\left(\delta_i^{sun} - T_{\ell sun,i}^n\right)\right] + g_{E,i}^{sha} \left[q_{sat}\left(T_{\ell sha,i}^n\right) + s_i^{sha}\left(\delta_i^{sha} - T_{\ell sha,i}^n\right)\right] \\ & + g_{a,i} q_{ref}^{n+1} \end{aligned} \tag{S23}$$

and other terms are as given before.

Special boundary conditions are also needed for the first layer ($i=1$), where $\theta_{i-1}^{n+1}=T_0^{n+1}$ and $q_{i-1}^{n+1}=q_0^{n+1}$ are the ground surface temperature and water vapor concentration, respectively, so that

$$a_{1,i}=0 \tag{S24}$$

$$b_{11,i}=\frac{\rho_m \Delta z_i}{\Delta t}+g_{a,i-1}+g_{a,i}+g_{H,i}^{sun}\left(1-\alpha_i^{sun}\right)+g_{H,i}^{sha}\left(1-\alpha_i^{sha}\right)-g_{a,i-1}\alpha_0 \tag{S25}$$

$$b_{12,i}=-g_{H,i}^{sun}\beta_i^{sun}-g_{H,i}^{sha}\beta_i^{sha}-g_{a,i-1}\beta_0 \tag{S26}$$

$$d_{1,i}=\frac{\rho_m \Delta z_i}{\Delta t}\theta_i^n+g_{H,i}^{sun}\delta_i^{sun}+g_{H,i}^{sha}\delta_i^{sha}+g_{a,i-1}\delta_0 \tag{S27}$$

$$a_{2,i}=0 \tag{S28}$$

$$b_{21,i}=-g_{E,i}^{sun}s_i^{sun}\alpha_i^{sun}-g_{E,i}^{sha}s_i^{sha}\alpha_i^{sha}-h_{s0}s_0 g_{s0}\alpha_0 \tag{S29}$$

$$b_{22,i}=\frac{\rho_m \Delta z_i}{\Delta t}+g_{s0}+g_{a,i}+g_{E,i}^{sun}\left(1-s_i^{sun}\beta_i^{sun}\right)+g_{E,i}^{sha}\left(1-s_i^{sha}\beta_i^{sha}\right)-h_{s0}s_0 g_{s0}\beta_0 \tag{S30}$$

$$\begin{aligned} d_{2,i}=&\frac{\rho_m \Delta z_i}{\Delta t}q_i^n+g_{E,i}^{sun}\left[q_{sat}\left(T_{\ell sun,i}^n\right)+s_i^{sun}\left(\delta_i^{sun}-T_{\ell sun,i}^n\right)\right]+g_{E,i}^{sha}\left[q_{sat}\left(T_{\ell sha,i}^n\right)+s_i^{sha}\left(\delta_i^{sha}-T_{\ell sha,i}^n\right)\right] \\ &+h_{s0}\left[q_{sat}\left(T_0^n\right)+s_0\left(\delta_0-T_0^n\right)\right]g_{s0} \end{aligned} \tag{S31}$$

and other terms are as given before.