A Review of
"On the numerical stability of surface-atmosphere coupling in weather and climate models"
by Anton Beljaars, Emanuel Dutra and Gianpaolo Balsamo

---

1 Introduction

This paper aims at addressing a stability problem found in the coupling between the atmospheric surface layer (ASL) and a snow/ice model. An instability associated to a synchrony problem between the snow/ice model and the surface flux computation is identified and illustrated by simple numerical experiments. A somehow empirical way to suppress this type of instability is introduced and numerically tested. The paper is structured as follows. In Sec. 2, the continuous problem is introduced, it consists of a vertical diffusion equation representing temperature evolution in the snow complemented with a flux (Neumann) surface boundary condition. At a discrete level, this diffusion equation is solved using an Euler backward scheme as usually done in state-of-the-art numerical models. Sec. 3 briefly describes the way to compute the surface flux for the snow model as a function of the atmospheric temperature. This computation is based on the standard semi-empirical Monin-Obukhov (MO) theory. Then, two ways of numerically evaluating this surface flux within an Euler backward framework are presented and referred to as *Explicit flux coupling* (which is the preferred way to proceed in numerical codes) and *Implicit flux coupling*. In Sec. 4, an idealized one-dimensional numerical experiment is used to illustrate the occurrence of a numerical instability when an explicit flux coupling is used jointly with a very fine vertical resolution at the top of the snow model. This instability is absent if an implicit flux coupling is used. In order to control this instability, an extrapolation in time based on physical parameters is presented in Sec. 5 to make a prediction of the surface flux at time level $n+1$ knowing the state variables of the snow model at time n. This ad-hoc extrapolation is built on scaling arguments to estimate the propagation of a surface perturbation over one time-step.

2 Overall quality of the paper

This paper raises a very interesting problem which occurs when a diffusion term is constrained by boundary conditions computed using a bulk formulation arising from the MO theory. In this case, instead of the classical Neumann or Dirichlet boundary condition we have to treat a linear combination of both (sometimes called Robin condition). Indeed, boundary condition (3) in the paper can be written as

$$K\partial_z T + \lambda_t T = g(t), \qquad \text{for } z = 0,$$

with $g(t) = \lambda_t T_a$. The paper clearly emphasizes the dilemma we get when discretizing this boundary condition since we expect $K\partial_z T(z=0)$ and $T(z=0)$ to be provided at the same moment in time which is at time level $n + 1$ if a backward Euler is used. If this is not the case, a numerical instability can occur even if an unconditionally stable implicit scheme is used to advance the diffusion term. This type of instability is generally unnoticed in the literature because it occurs under very unusual situations. Just for raising this issue and trying to circumvent it, this paper should be considered for publication. The paper is well written and the simple numerical experiments are nicely chosen to illustrate the punchline of the paper. **However I recommend major revisions to make the paper less misleading and more convincing because this issue is important for the modeling community**. The following points must be addressed, because as is the paper has a lack of arguments/proofs of numerical nature. To strengthen the message, I personally think that those proofs should be given in this paper and not in a separate paper with possibly different authors.

3 General comments

Some of the comments made in this section are supported by additional material presented in Appendix. This goes beyond the normal reviewer duty but the aim is to illustrate how simple the stability analysis could be to convincingly support the various ideas introduced in the paper.

- The manuscript considers an instability of numerical nature, in this regard we expect a stability analysis to characterize under what circonstances the instability can occur. Following Lemarié et al. (2015) or more simply App. A below, this type of stability analysis is technically affordable and does not require too much theoretical efforts. An important dimensionless number is

$$\gamma = \frac{\lambda_t \Delta t}{\Delta z (\rho C)}$$

  which must stay small enough to ensure stability of the coupling between the ASL and the various type of surfaces. Of course all the hypothesis of the Von Neumann stability analysis are not met in this particular problem but this type of analysis always provide useful hints. It could also be interesting to provide in the paper some typical values of the parameter γ depending on the surface model (ocean, snow, vegetation, etc) so that the reader can assess how specific to the ASL/snow coupling this problem is. Is it standard to use a vertical resolution Δz of the order of 10^{-3} m in snow models ?
- The paper could leave the impression that the temporal variation of the atmospheric temperature T_a plays a role in the development of the instability. However it must be clear that the instability occurs even if the atmospheric temperature is held constant in time or is simply set to zero. Hence, this instability can occur in coupled models but also in uncoupled models forced with a bulk formulation.
- The statement in the abstract "*These (instabilities) are due to the choice of large integration time-step, aiming at reducing computational burden*" must be mitigated because it is not the only contributing factor, the vertical resolution or the transfer coefficients value are other important parameters.
- p. 6 line 5, it is adventurous to draw any conclusion on the accuracy of the proposed method based solely on the simple numerical experiments presented in the paper without any well established metric like an $\mathcal{L}^2$ or an $\mathcal{L}^\infty$ norm and a convergence study with spatial/temporal resolution. In this paper, the emphasis is on stability and the current experiments or mathematical analysis do not allow any conclusion on accuracy.
- It is not rigorous enough to assess the efficiency of the proposed empirical coupling method based only on an idealized numerical experiment under very specific conditions. For example, in App. B it is shown that, indeed, the proposed method allows for an unconditionally stable coupling between the ASL and the surface model whatever the parameter values.
- Since this paper is considered for publication in GMD, it would be worthwhile to provide additional

details about the implementation of the proposed method in a numerical model with non-uniform grid and flow-dependent diffusion coefficients.

- In the conclusion, it could be interesting to give some comments on the expected benefits of your approach in realistic models. Besides stability, do you expect significant differences in the physical solutions ?

4 Technical corrections

- The way to specify units is inconsistant throughout the paper. For example, units are missing for Δz and Δt in p. 5 (lines 9 to 20), sometimes units are in italic sometimes not (e.g. p. 9, lines 15-16) ...
- In eqn (19) it should be T_0 and not T_0^n
- p. 7 line 9, it should be $\Delta z \ll \delta$ and not $\Delta z \ll \delta z$
- In Figure 1, T_{sk} and T_a could be added (instead of T_{10} which is never used in the paper). λ_{sk} and λ_a could also be reported on the figure.
- In figure 2 the left panels show the skin temperature T_{sk} whereas the left panels of Figure 5 show T_1. To facilitate the comparison, the same quantity should be plotted.
- Appendix A is relatively trivial and does not provide useful informations. It could be interesting to use this appendix to be more specific about the elimination and back-substitution steps when solving the tridiagonal problem. We guess a Thomas algorithm is used but it is not explicitly stated.

A Stability analysis of the explicit flux coupling

In the following we consider that $T_a = 0$. It must be clear that this choice does not affect the stability analysis. For personal convenience, we consider here that the vertical grid goes from $k = 1$ at the bottom of the snow model to $k = N$ at the air-snow interface (T_N is thus equivalent to T_1 in the paper). Adapting eqn (6) in the paper we obtain

$$T_N^{n+1} = T_N^n - \frac{\Delta t}{\Delta z(\rho C)}\left[\left(\lambda_t T_N^n + \frac{K}{\Delta z}T_N^{n+1}\right) - \left(\frac{K}{\Delta z}T_{N-1}^{n+1}\right)\right]$$

for a constant diffusion K and grid spacing Δz. Introducing the dimensionless coefficients

$$\gamma = \frac{\lambda_t \Delta t}{\Delta z(\rho C)}, \qquad \sigma = \frac{K\Delta t}{\Delta z^2(\rho C)}$$

where σ is the standard parabolic (diffusion) Courant number, we end up with the equivalent form

$$T_N^{n+1} = T_N^n - \gamma T_N^n - \sigma T_N^{n+1} + \sigma T_{N-1}^{n+1}. \tag{A.1}$$

Assuming that $T_k^{n+1} = \widehat{T}^{n+1}e^{-ik\theta}$ with $\theta = k_z\Delta z \in [0,\pi]$ a normalized wavenumber in the vertical direction, the $\widehat{T}$'s satisfy

$$\widehat{T}^{n+1} = \widehat{T}^n - \gamma\widehat{T}^n - \sigma\widehat{T}^{n+1} + \sigma\widehat{T}^{n+1}e^{-i\theta},$$

which subsequently provides the following amplification factor $\mathcal{A}$

$$\mathcal{A} = \frac{1-\gamma}{1+\sigma(1-e^{-i\theta})}.$$

The modulus of the amplification factor is less than one as long as

$$\gamma \leq 1 + \sqrt{1 + 2\sigma(1+\sigma)(1-\cos\theta)} \qquad \Rightarrow \gamma \leq 2, \text{ for } \theta \in [0;\pi]$$

which corresponds to the stability condition of the explicit flux coupling. Note that in general this stability limit corresponds to a conservative condition and we can expect the model to be stable with larger values of γ because of the regularizing effect of diffusion below the surface and of the atmospheric response.

B Stability analysis of the empirical flux coupling

Using Eqns (14) and (22) in the paper, we easily find that in the empirical coupling the term γT_N^n in (A.1) is replaced by $\frac{\gamma T_N^n}{1+\alpha\lambda_t}$ with $\alpha = f(\sqrt{\sigma})\sqrt{\frac{\Delta t}{K(\rho C)}}$. Interestingly enough, if f is such that $f(\sqrt{\sigma}) = \sqrt{\sigma}$ we obtain with this modified formulation that $\frac{\gamma T_N^n}{1+\alpha\lambda_t} = \frac{\gamma T_N^n}{1+\gamma}$. This means that in the case $\Delta z \gg \delta$ (i.e. $f(x) = x$) the amplification factor is

$$\mathcal{A} = \frac{1 - \gamma/(1+\gamma)}{1 + \sigma(1 - e^{-i\theta})}$$

whose modulus is always smaller than 1 thus indicating unconditional stability.

References

Lemarié, F., Blayo, E., Debreu, L., 2015. Analysis of ocean-atmosphere coupling algorithms: Consistency and stability. Procedia Computer Science 51(0), 2066 – 2075.