The authors thank both referees for the extensive reviews, constructive comments, and for providing many suggestions for improvement. The practical relevance of this paper is clearly recognized and we appreciate the encouragement to publish after revision.

Revisions suggested by reviewer 1

Introduction

... a numerical instability can occur even if an unconditionally stable implicit scheme is used to advance the diffusion term. This type of instability is generally unnoticed in the literature because it occurs under very unusual situations. Just for raising this issue and trying to circumvent it, this paper should be considered for publication. The paper is well written and the simple numerical experiments are nicely chosen to illustrate the punchline of the paper. However I recommend major revisions to make the paper less misleading and more convincing because this issue is important for the modeling community. The following points must be addressed, because as is the paper has a lack of arguments/proofs of numerical nature. To strengthen the message, I personally think that those proofs should be given in this paper and not in a separate paper with possibly different authors.

The manuscript has been revised in a major way by addressing the misleading aspects, and by adding a stability analysis in Appendix B.

General comments, bullet 1

The manuscript considers an instability of numerical nature, in this regard we expect a stability analysis to characterize under what circonstances the instability can occur.

It could also be interesting to provide in the paper some typical values of the parameter γ.

Is it standard to use a vertical resolution Δz of the order of 0.001 m in snow models ?

Reviewer 1 recognizes the practical relevance of the paper, but feels that a formal stability analysis is missing. We agree that a stability analysis is a welcome addition to the paper. For this purpose a "matrix stability analysis" has been added as Appendix B. It confirms that the implicit coupling and the parametrized implicit coupling are unconditionally stable.

A new table 2 with values for γ and σ has been included.

Snow models with 0.001 m vertical resolution are rare, but models must be able to cope with such thin layers, e.g. during the early accumulation of snow and the final melt. Reference is made now to such situations in the introduction.

General comments, bullet 2

The paper could leave the impression that the temporal variation of the atmospheric temperature Ta plays a role in the development of the instability. However it must be clear that the instability occurs even if the atmospheric temperature is held constant in time or is simply set to zero. Hence, this instability can occur in coupled models but also in uncoupled models forced with a bulk formulation.

With the stability analysis of Appendix B, it is clear now that numerical stability does not depend on the forcing, because it assumes zero perturbations at the forcing level.

General comments, bullet 3

The statement in the abstract "These (instabilities) are due to the choice of large integration time-step, aiming at reducing computational burden" must be mitigated because it is not the only contributing factor, the vertical resolution or the transfer coefficients value are other important parameters.

We agree that the statement about long time steps is not very precise. It is of course about the time step in relation to the physical time scale of the discretized problem (which depends on vertical discretization and properties of the medium). The abstract has been revised accordingly.

General comments, bullet 4

p. 6 line 5, it is adventurous to draw any conclusion on the accuracy of the proposed method based solely on the simple numerical experiments presented in the paper ....

The statement on p.6 line 5 about accuracy is indeed not very general and limited to the case that is presented. A comment of this nature has been added at the beginning of the previous paragraph.

General comments, bullet 5

It is not rigorous enough to assess the efficiency of the proposed empirical coupling method based only on an idealized numerical experiment under very specific conditions.

We agree that conclusions on stability can not be generalized on the basis of limited numerical experimentation. A stability analysis has been added in Appendix B, to which reference is made at various places in the manuscript.

General comments, bullet 6

Since this paper is considered for publication in GMD, it would be worthwhile to provide additional details about the implementation of the proposed method in a numerical model with non-uniform grid and flow-dependent diffusion coefficients.

We agree that it is of interest to consider a non-uniform grid and flow dependent diffusion coefficients. The concluding section already discusses the case of non-uniform diffusion coefficients, and in fact the aerodynamic coupling between atmosphere and snow can just be seen as an extreme jump in properties of the medium. This point has been added in the discussion.

Temperature dependent diffusion coefficients is a completely different story because it potentially introduces a nonlinear instability which is classic in atmospheric boundary layer schemes (see e.g. Kalnay and Kanamitsu 1988: "Time schemes for strongly nonlinear damping equations", Monthly Weather Review, 116, 1945-1958). Discussion of this issue is beyond the scope of the paper.

General comments, bullet 7

In the conclusion, it could be interesting to give some comments on the expected benefits of your approach in realistic models. Besides stability, do you expect significant differences in the physical solutions ?

The benefit for real models is stability (which is an absolute requirement) and advantage can be taken from finer vertical discretization resulting in a faster response of the surface temperature. The first paragraph of the concluding section has been rewritten.

Technical corrections, bullet 1

The way to specify units is inconsistent throughout the paper.

Units are italic now throughout.

Technical corrections, bullet 2

In eqn (19) it should be T_0 and not T_0^n

Equation (19) has been deleted to improve clarity in response to reviewer 2.

Technical corrections, bullet 3

p. 7 line 9, it should be $\Delta z << \delta$ and not $\Delta z << \delta z$

Has been corrected.

Technical corrections, bullet 4

In Figure 1, T_{sk} and T_a could be added (instead of T_{10} which is never used in the paper). λ_{sk} and λ_a could also be reported on the figure.

Figure 1 has been adapted.

Technical corrections, bullet 5

In figure 2 the left panels show the skin temperature T_{sk} whereas the left panels of Figure 5 show T_1. To facilitate the comparison, the same quantity should be plotted.

Fig. 5 shows T_{sk} now.

Technical corrections, bullet 6

Appendix A is relatively trivial and does not provide useful informations. It could be interesting to use this appendix to be more specific about the elimination and back-substitution steps when solving the tridiagonal problem. We guess a Thomas algorithm is used but it is not explicitly stated.

A paragraph has been added in appendix A to explain the tri-diagonal solver. It is a standard Gaussian elimination procedure, which I think, is also called the Thomas algorithm.

Revisions suggested by reviewer 2

Introduction

I found this study reasonably well argued and set out, with clear additional evidence provided from one-dimensional simulations, and am persuaded that the method described provides a good approximation to a semi-implicit scheme, with significant improvements over a traditional explicit scheme. The study makes a useful addition to the literature. My only comments relate to ways in which the method might perhaps be explained more clearly and consistently. In particular, I am not sure that the algorithm can be described as acting like a "fully" implicit coupling scheme (see point 2).

The suggestion to improve clarity were very helpful. The different points are described below.

1. **Page 3, line 7: The matrix equations can be solved by successive elimination of the C-coefficients from the bottom upward.**

   This is a crucial step as it provides the initial linear relation between surface flux and top layer temperature; however, I had to work through it quite carefully to understand how this produced equation (8). It is also a little confusingly written as strictly speaking it is the variables that are eliminated, not the coefficients. I wonder if it would be worth expanding this line to demonstrate the elimination of bottom layer temperature from the lowest pair of equations, and its result

   The solution of the tri-diagonal matrix by Gaussian elimination is fairly standard. The details of the elimination process have been added in Appendix A.

2. **Page 4, line 18: "Together with Eq.(13) T_1^{n+1} and G_0 can be computed:" (and following equations (14,(15)).**

   It is probably a trivial point, but the solution of equations (14) and (15) actually depends on the future air temperature, T_a^{n+1}, already being known. I would guess that it is assumed in the method that the atmospheric simulation is solved first, using bottom boundary conditions from the previous time step, hence T_a^{n+1} is known when the time comes to calculate the surface flux.

   The availability of the forcing temperature at the new time level is a simplification for this study. However, it is not a limitation. The coupling scheme to an atmospheric model (i.e. not just a forcing level) as described by the reviewer, is not what we have in mind. Although it is not the topic of the paper, it is obviously a shortcoming of the paper not to discuss it. Thanks for raising this issue.

The way it can be done in a fully implicit way is by doing the elimination phase of the tri-diagonal matrix for turbulent diffusion in the atmosphere (from top to surface) exactly in same way as is done for the surface heat diffusion in this paper. This procedure leads also to a linear relation between temperature at the lowest atmospheric model level and the heat flux into the surface. This relation replaces the imposed temperature at the new time level. This is precisely the coupling procedure followed in the ECMWF model and is compatible with the Best et al. (2004) approach. It is equivalent to solving a single tri-diagonal matrix that handles the entire atmosphere and surface as a single implicit problem.

A paragraph has been added in the paper to discuss how the method can be applied in a fully coupled atmosphere / surface model.

3. Page 6, line 30 - page 7, line 8 (discussion of scaling behavior of surface temperature evolution).

This section (Section 5) contains the central idea of the study, which is impressive in its simplicity. The idea is based on the fact that dimensional constraints on solutions to the diffusion equation mean that in the case of an initially isothermal profile, the evolution of the surface temperature in response to any surface flux must take a very specific form, described in equation (18) of Beljaars et al. My issue is that having derived equation (18), the authors appear to digress into a discussion of the surface temperature simulation before arriving at equation (22), which describes the form the coefficient must take. I think that this is unnecessary (see below). It also appears to conflate two different concepts: the surface skin temperature (SSKT) and top layer temperature (TLT). It is the TLT that the authors are trying to approximate using the scaling arguments, but in equations (19) and (20) they effectively derive the form would take if they were instead trying to approximate the SSKT. The authors then state that this form must also take account of Δz for an approximation of TLT, due to the system being discretized. But I think that the form must take account of Δz for the (simpler) reason that it is TLT they are trying to approximate, not SSKT. It is possible that the authors view TLT as being the "discrete" approximation to SSKT, but I think that it is much more accurate to view it as the discrete approximation to temperature at a depth of $\Delta z/2$

The reviewer raises an interesting point on the temporal evolution of temperature as the result of a perturbation at the surface. It is argued that the final scaling relation can be achieved in a more logical way. Reading the manuscript again, we agree that the transition from the continuous to the discrete problem is vague, which does not help clarity of the manuscript.

The simplest way of deriving a similarity relation for α is by doing a basic dimension analysis, which leads directly to equation (19) of the revised manuscript for the continuous problem (given time scale Δt). The discrete problem adds a new length scale resulting in a functional dependence on $(\delta/\Delta z)$, i.e. equation (21) of the revised manuscript. We opted to start with the traditional scaling relation for the diffusion equation and then to add the complexity of the discretization.

The derivation by the reviewer is attractive, but after thinking about it more carefully, I am not sure any more. The attractive aspect is the simplicity and the fact that function h (eq. 18 of the manuscripts) comes back as function f (eq. 21). It is important to realize that the meaning is not necessarily the same. Function h represents the shape of the temperature profile, whereas function f is an empirical function that describes the transition from one asymptotic scaling regime (the non-discretized problem) to another (the extremely coarse vertical resolution regime). The two functions are related and perhaps even close, but I am not sure they are the same. To demonstrate the potential difference, it is necessary to consider the average temperature (or average h) over the discretization intervals, instead of using midpoint values, otherwise conservation is lost (I think).

The beauty of similarity theory is that we don't need to answer this question. Function f is an empirical function and we can derive it from numerical experiments.

In view of the discussion above, the manuscript has been modified such that it is clear where the transition is made from the continuous to the discrete problem.

On the numerical stability of surface-atmosphere coupling in weather and climate models

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Abstract. Coupling the atmosphere with the underlying surface presents numerical stability challenges in cost-effective model integrations used for operational weather prediction or climate simulations. These are due to the choice of large integration time-step compared to the physical time scale of the problem, aiming at reducing computational burden, and to an explicit flux coupling formulation, often preferred for its simplicity and modularity. ~~The atmospheric~~ Atmospheric models therefore use the surface-layer temperatures (representative of the uppermost soil, snow, ice, water, etc.,) at the previous integration time-step in all surface-atmosphere heat-flux calculations and prescribe fluxes to be used in the surface models' integrations. Although both models may use implicit formulations for the time stepping, the explicit flux coupling can still lead to instabilities.

In this study, idealized simulations with a fully coupled implicit system are performed to derive an empirical relation between surface heat flux and surface temperature at the new time level. Such a relation mimics the fully implicit formulation by allowing to estimate the surface temperature at the new time level without solving the surface heat diffusion problem. It is based on similarity reasoning and applies to any medium with constant heat diffusion and heat capacity parameters. The advantage is that modularity of the code is maintained and that the heat flux can be computed in the atmospheric model in such a way that instabilities in the snow or ice code are avoided. Applicability to snow/ice/soil models with variable density is discussed, and the loss of accuracy turns out to be small. A formal stability analysis confirms that the parametrized implicit flux coupling is unconditionally stable.

1 Introduction

Coupling atmospheric models to the underlying surface model, involves both scientific and technical issues. Models of the atmospheric circulation tend to be computer intensive and therefore often employ long time steps (up to one hour), which is a challenge for stability and accuracy ~~(Beljaars et al., 2004)~~ (Beljaars et al., 2004; Lemarié et al., 2015) . The turbulent diffusion part of these codes provides the coupling to the surface, has short physical time scales near the surface and therefore needs implicit numerics for stability. The surface may be vegetation, soil, snow, ice, or a combination of these in a tile scheme. Best et al. (2004) propose a coupling strategy to the surface that has a clean interface between atmosphere and surface code, and allows to include the surface or the top part of the surface in the implicit computations. This is often necessary for stability if the physical time scale of e.g. vegetation, soil, snow or ice surface is short compared to the model time step.

The ideal solution for stability is to combine the boundary layer heat diffusion and e.g. the snow or ice layer diffusion in a single implicit solver. However, modularity of the code and the complication of additional processes like phase changes and water percolation make this less practical. The standard solution is to compute fluxes at the surface on the basis of the old time level surface temperature. It is often called "explicit flux coupling". To improve stability and accuracy, West et al. (2016) recently proposed to move the flux coupling level one level down i.e. just below the surface. This has the advantage of including the fast responding surface layer in the fully implicit computations, which is beneficial for stability and accuracy.

Ongoing work at ECMWF on snow modelling raised similar issues. The existing single layer snow model (see e.g. Dutra et al., 2010), has already a minor stability issue when the snow layer becomes very thin, e.g. during the first snowfall in the season and at the final melt. This was addressed by introducing some empirical implicitness in the coupling by making an educated guess of the future snow temperature. Initial experimentation with a multilayer snow model (Dutra et al., 2012) showed even more frequent instabilities, so more implicitness in the coupling is required for stability.

In this paper, we propose a solution, that has the simplicity and modularity of the explicit flux coupling, but still has the stability of the fully implicit system. To derive simple solutions, the fully implicit coupled system is used as a reference. It is shown that the tri-diagonal set of equations corresponding to the discretized diffusion equation (for snow, ice or soil) can be converted to a relation between temperature and heat flux at the surface. The coefficients in this relation are then parametrized dependent on properties of the medium, time step and vertical discretization. The coefficients are put in dimensionless form, which makes the empirical coefficients universal and applicable to any medium and any discretization.

The experimental environment in this paper, is a simple model of a near surface air layer coupled to a snow pack by turbulent exchange. The atmosphere (e.g. at a height of 10 m, typical for atmospheric models) is assumed to have a diurnal cycle, and the response of temperature in the snow pack is considered. Although the following sections refer to snow only, the dimensionless framework ensures that the outcome is valid for any medium.

The following two sections (2 and 3) describe the equations for the discretized snow layer and the turbulent coupling between atmosphere and snow. Sections 4, 5 and 6 describe the numerical solution for an idealized diurnal cycle, the parametrization of the coefficients that relate heat flux and top layer snow temperature and the testing of the proposed scheme. Finally, the results and their applicability are briefly discussed in the concluding section. Also the implications of non-uniform snow density are discussed. The numerical solver and a formal stability analysis are described in Appendix A and B respectively.

2 Implicit numerical solution of the diffusion equation

We consider the diffusion equation for temperature in snow

$$\rho C \frac{\partial T}{\partial t} = \frac{\partial G}{\partial z} \,, \tag{1}$$

$$G = K \frac{\partial T}{\partial z} \,, \tag{2}$$

where $\rho\ (kg\,m^{-3})$ is density, $C\ (J\,kg^{-1}K^{-1})$ is heat capacity, $T\ (K)$ is temperature, $G\ (W\,m^{-2})$ is heat flux, and $K\ (W\,m^{-1}K^{-1})$ is the diffusion coefficient for heat. The boundary conditions are:

$$G \;=\; G_0 \ for\ z = 0\,, \tag{3}$$

$$G \;=\; 0 \ for\ z \to -\infty\,. \tag{4}$$

For numerical stability with long time steps it is necessary to use an implicit scheme. With a vertical grid defined as in Fig. 1, the equation can be discretized as follows

$$(\rho C)j\frac{T_j^{n+1} - T_j^n}{\Delta t} = \frac{1}{\Delta z_j}\left(K_{j-1/2}\frac{T_{j-1}^{n+1} - T_j^{n+1}}{\Delta z_{j-1/2}} - K_{j+1/2}\frac{T_j^{n+1} - T_{j+1}^{n+1}}{\Delta z_{j+1/2}}\right)\,, \tag{5}$$

with boundary conditions

$$(\rho C)_1\frac{T_1^{n+1} - T_1^n}{\Delta t} = \frac{1}{\Delta z_1}\left(G_0 - K_{1+1/2}\frac{T_1^{n+1} - T_2^{n+1}}{\Delta z_{1+1/2}}\right)\,, \tag{6}$$

$$(\rho C)_{NL}\frac{T_{NL}^{n+1} - T_{NL}^n}{\Delta t} = -\frac{1}{\Delta z_{NL}}\left(K_{NL-1/2}\frac{T_{NL-1}^{n+1} - T_{NL}^{n+1}}{\Delta z_{NL-1/2}}\right)\,. \tag{7}$$

This set of equations forms a tri-diagonal system, with diagonals A, B and C (the coefficients are defined in Appendix ~~A~~ A). The matrix equations can be solved by successive elimination ~~of the C-coefficients~~ from the bottom upward such that the C-coefficients are replaced by zeros. At the same time, the equations are scaled ~~such that the~~ to obtain B-coefficients ~~become~~ that are equal to 1. Arriving at the top, it provides a solution for T_1^{n+1}. The solution for the other layers can be found by successive back-substitution of the temperatures going from top to bottom (see Appendix A for more details).

In case G_0 is not known, the elimination provides a linear relation between G_0 and T_1^{n+1}

$$T_1^{n+1} = \alpha G_0 + \beta\,. \tag{8}$$

This relation can be used to achieve fully implicit coupling with the air/surface interaction formulation.

3   Coupling to the lowest model level of the atmosphere

~~The heat flux~~ To focus on stability of the atmosphere surface coupling, it is assumed that the evolution of the near atmospheric temperature is known, e.g. as in standalone simulations of the land surface. However, this is not a limitation in full 3D models that typically use an implicit solver for the turbulent diffusion. In that case the atmospheric model will perform the downward elimination process (the same way as described in Appendix B). The result is a linear relation between the $n+1$ temperature at the lowest atmospheric level and the surface heat flux, which can be used with the air/land interaction formulae described below to achieve fully implicit coupling.

With a prescribed air temperature, the heat flux into the snow layer can be related to the air / surface temperature difference in the following way

$$G_0 = \rho_a c_p C_H |U|(T_a - T_{sk})\,, \tag{9}$$

where G_0 is the heat flux into the snow pack, ρ_a is air density, c_p is air heat capacity, C_H is the transfer coefficient between the atmospheric level and the surface, $|U|$ is absolute wind speed, T_a is air temperature, and T_{sk} is temperature of the snow surface (skin temperature).

The coupling through a transfer coefficient is standard and represents the integral profile function according to Monin Obukhov (MO) similarity (see e.g. Brutsaert, 1982). The transfer coefficient in neutral conditions is related to the height of the atmospheric level, and the surface roughness lengths of momentum and heat

$$C_H = \frac{\kappa^2}{ln(z_a/z_{om})ln(z_a/z_{oh})} \,, \tag{10}$$

where κ is the VonKarman constant (0.4), z_a is the height of the atmospheric level, z_{om} is the surface roughness length for momentum, and z_{oh} is the surface roughness length for heat. Stability can be included by extending the logarithmic terms with the integral MO stability functions.

In the vertically discretized snow (see Fig. 1), the temperature of layer 1 is assumed to be at the midpoint which is different from the skin temperature. Therefore, the total conductivity between the atmosphere and the first snow layer (λ_t) is composed of two components: the turbulent transfer in the air above the surface (λ_a) and the conductivity of half of the top snow layer (λ_{sk}). The two conductivities are in parallel, because the inverse of conductivities (resistances) are in series, leading to the following formulation for the heat flux into the snow

$$G_0 \;=\; \lambda_t(T_a - T_1) \,, \tag{11}$$

with

$$\lambda_t \;=\; \frac{\lambda_a \lambda_{sk}}{\lambda_a + \lambda_{sk}} \,,$$
$$\lambda_a \;=\; \rho_a c_p C_H |U| \,,$$
$$\lambda_{sk} \;=\; \frac{2\,K_{1-1/2}}{\Delta z_1} \,.$$

Two different time stepping procedures are considered:

i. **Explicit flux coupling**. This is the traditional approach where the expression for the surface flux uses the previous time level of the surface temperature leading to the following discretization of ~~Eq.~~ equation (11)

$$G_0 \;=\; \lambda_t(T_a^{n+1} - T_1^n) \,. \tag{12}$$

With the explicit specification of the flux at the surface flux, the tridiagonal system can be solved directly.

ii. **Implicit flux coupling.** The discretization of ~~Eq.~~ equation (11) reads

$$G_0 \;=\; \lambda_t(T_a^{n+1} - T_1^{n+1}) \,, \tag{13}$$

With this fully implicit formulation, the surface heat flux can not be specified explicitly, so it has to be found as part of the coupled atmosphere/surface system. For that purpose the tri-diagonal problem is solved in two steps. First, the

elimination part is performed resulting in a solution for α and β in ~~Eq.~~ equation (8). Together with ~~Eq.~~ equation (13), T_1^{n+1} and G_0 can be computed:

$$T_1^{n+1} = \frac{\alpha\lambda_t T_a^{n+1} + \beta}{1 + \alpha\lambda_t} , \tag{14}$$

$$G_0 = \frac{\lambda_t(T_a^{n+1} - \beta)}{1 + \alpha\lambda_t} . \tag{15}$$

5 Finally the entire temperature profile can be resolved by performing the back-substitution in the tri-diagonal solver.

4 Solutions with a simple multilayer snow model

In this section, solutions are considered for a 1 ~~m~~ m thick snow layer with constant heat capacity and heat diffusion coefficients. Idealized temperature forcing from the atmosphere is prescribed as a sinusoidal diurnal cycle. The choice of constants is documented in Table 1. ~~List of parameters used in the idealized simulation of a snow layer Parameter Description Value~~
10 ~~Units ρ snow density 150 $kg\,m^{-3}$ ρ_{ice} ice density 920 $kg\,m^{-3}$ C snow (and ice) heat capacity 2228 $J\,kg^{-1}K^{-1}$ K_{ice} ice heat diffusion coefficient 2.2 $W\,m^{-1}K^{-1}$ K snow heat diffusion coefficient $K_{ice}(\rho/\rho_{ice})^{1.88} W\,m^{-1}K^{-1}$ ρ_a air density 1.2 $kg\,m^{-3}$ c_p air heat capacity 1005 $J\,kg^{-1}K^{-1}$ $|U|$ absolute wind speed 4 $m\,s^{-1}$ z_{om} roughness length for momentum 0.0001 m z_{oh} roughness length for heat 0.0001 m z_a height atmospheric forcing level 10 m κ VonKarman constant 0.4 — D total depth of snow layer 1 m~~ The initial temperature profile at $t = 0$ is set to -5^oC, and a single sinusoidal diurnal cycle with an
15 amplitude of 1^oC is imposed at the ~~10m~~ 10 m level in the atmosphere

$$T_{10} = -5 + sin(\frac{2\pi t}{3600.*24.}) . \tag{16}$$

The simulations are performed with different uniform vertical discretizations and different time steps. Fig. 2 shows time series of the snow skin temperature (left column) and the ground heat flux (right column), with the two schemes. The fairly long time step of 3600 seconds is selected to illustrate stability and time truncation issues, and a short time step of 100 seconds for
20 comparison. In the latter case time truncation errors are small for both schemes (convergence was verified). The three rows in Fig. 2 are for different vertical discretizations: 0.2, 0.02 and 0.002 ~~m~~ m.

The first thing to note is that amplitude and phase of the skin temperature diurnal cycle only have a small dependence on vertical resolution. This is surprising because the amplitude of diurnal cycle of layer 1 with ~~$\Delta z = 0.2$~~ $\Delta z = 0.2\,m$ is only 20% of the amplitude with ~~$\Delta z = 0.02$~~ $\Delta z = 0.02\,m$. The reason that the skin temperature is still reasonable is due to
25 the conductivity between the middle of the layer and the top (much lower with ~~$\Delta z = 0.2$ than with $\Delta z = 0.02$~~ $\Delta z = 0.2\,m$ than with $\Delta z = 0.02\,m$). So at low vertical resolution, a substantial part of the temperature signal at the snow skin is due to the "interpolation" between air and middle of the first snow layer making use of the air conductivity (λ_a) and the snow conductivity of half the top layer (λ_{sk}). One might interpret this result as a justification for rather low vertical resolution. However, it should be realized that the forcing has the diurnal time scale only. With faster time scales e.g. due to moving clouds and frontal
30 passages, a relatively thick near surface layer will not be able to respond.

5

The second result is Although it is impossible to draw general conclusions about accuracy from limited experimentation, we note that the fully implicit solution with $\Delta t = 3600$ $\Delta t = 3600\,s$ is very close to the short time step solution with $\Delta t = 100$ $\Delta t = 100\,s$, so the long time step does not compromise accuracy in this case, although the time stepping is first order accurate only. However, the solution with explicit coupling deviates visibly from the implicit and very short time step solutions (compare the red solid curve in middle/left panel of Fig. 2 with the blue curve). Apparently, it is the mismatch of time levels in the flux computation that is detrimental to accuracy. The error is particularly visible as a phase error.

Finally, the explicit coupling turns out to be unstable for very thin snow layers (see lower panels in Fig. 2 for $\Delta z = 0.002$. Also for this case the long time step solution with implicit coupling is fairly accurate as it is very close to the short time step solution. These experimental results are confirmed by a formal stability analysis in Appendix B. The explicit flux coupling is unstable for a particular parameter range and the implicit flux coupling is unconditionally stable.

Because of the good stability and accuracy characteristics, we develop in the next section a parametric form of α and β in Eq. equation (8).

5 Scaling relations for α and β

As suggested above, it is desirable to have all the flux formulations (also for the atmosphere/surface exchange) at the new time level $n+1$. This implies the fully implicit option as described in sections 2 and 3. It also requires to perform the elimination part of the tri-diagonal solver to find the relation between T_1^{n+1} and G_0 according to Eq. equation (8). Because of code modularity it is desirable to make a reasonable estimate of the heat flux into the snow, before the snow code is executed. Therefore, an educated guess is made of the coefficients α and β in Eq. equation (8) without solving the tri-diagonal system, i.e. α and β are parameterized parametrized.

For that purpose, we make use of similarity theory for the diffusion equation with constant coefficients. If we think of an infinite medium (thick snow layer) with uniform temperature T_o and make a jump at the surface to T_{new} at $t = 0$, we have to consider the following basic variables: temperature T the temperature change $T - T_0$ at time t, $T_o, T_{new}, T_{new} - T_0,$ $K/(\rho C)$, and depth z. According to the Buckingham Pi Theorem (Stull, 1988), 5 variables with 3 dimensions (m, s, and K), lead to two independent dimensionless groups: $(T - T_0)/(T_{new} - T_0)$ and z/δ, where

$$\delta = \left(\frac{Kt}{\rho C}\right)^{1/2}. \tag{17}$$

Length scale δ is the natural length scale of the medium for time scale t after which the temperature change at the surface was applied. From the physical point of view, δ is the typical depth to which the perturbation of the surface temperature has propagated at time t. The implication is that $(T - T_0)/(T_{new} - T_0)$ is a universal function of z/δ. At this stage we do not care about the form, although the solution can be easily found found easily by transforming the equation to the new coordinate z/δ, which allows to separate the time dependence and the depth dependence leading to an ordinary differential equations which can be solved analytically (Carslaw and Jaeger, 1959).

Similarly, we can apply an external forcing by suddenly applying a heat flux G_0 at time 0 and look for the temperature response. Instead of scaling the temperature with ~~$T_{new} - T_0$, we convert~~ the temperature jump, we make the temperature change dimensionless with G_0 ~~into a temperature scale~~ and obtain

$$\frac{K(T - T_0)}{\delta G_0} = h\left(\frac{z}{\delta}\right) \text{ , or } T = \frac{\delta G_0}{K} h\left(\frac{z}{\delta}\right) + T_0 \text{ ,} \tag{18}$$

where h is a universal function. ~~If we are interested in the surface temperature only (i.e.~~ For $z = 0$ ~~), the left hand side becomes a constant, which we will call h_0 (which is order 1) .~~

~~This line of reasoning can also be applied to the evolution of the surface temperature during a single time step of the diffusion problem with discrete equations. Equation (18) can be written as Eq. (8), with $t = \Delta t$, $T = T_1^{n+1}$, $T_0 = T_1^n$, and Eq. (17)~~ equation (18) is of the form of equation (8). With time scale Δt and substitution of the expression for δ, ~~resulting in Therefore we~~ we therefore expect the following scaling behavior for α

$$\alpha \sim \left(\frac{\Delta t}{K \rho C}\right)^{1/2} \text{ .} \tag{19}$$

It indicates the surface temperature response to a 1 W/m^2 heat flux forcing over a finite time step Δt.

The scaling arguments above apply to the continuous system. For the discretized system, the scaling behavior of α also depends on Δz, which introduces a dependence on the dimensionless variable $\Delta z/\delta$. For a very fine grid (~~$\Delta z << \delta z$~~ $\Delta z << \delta$), the discrete system behaves like the continuous system and ~~Eq.~~ equation (19) applies. For a very thick top layer ($\Delta z >> \delta$), the heat flux is simply distributed over the top layer and the following applies

$$\alpha = \frac{\Delta t}{\Delta z \rho C} \text{ .} \tag{20}$$

In general the dimensionless α should be a universal function of $\delta/\Delta z$, i.e.

$$\alpha \left(\frac{K \rho C}{\Delta t}\right)^{1/2} = f\left(\frac{\delta}{\Delta z}\right) = f\left(\frac{(K\Delta t)^{1/2}}{\Delta z (\rho C)^{1/2}}\right) \text{ .} \tag{21}$$

The empirical function can be "measured" by running the numerical model as in the previous section for a range of time steps and vertical discretizations. Note that α remains constant during the time stepping and does not depend on the temperature profile. It is just a property of the tri-diagonal matrix which only contains properties of the medium, the time step and the level thickness. The results are shown in Fig. 3. Time steps range from 100 ~~s~~ s to 3600 ~~s~~s, and layer thicknesses are used from 0.002 ~~m~~m to 0.2 ~~m~~m, with a total snow depth of 1 ~~m~~m for all simulations

For small ratios of $\delta/\Delta z$, the universal function should scale with ~~Eq.~~ equation (20) and for large values with (19). Surprisingly, coefficient h_0 turns out to be 1. An empirical fit is proposed that makes a smooth transition between the two regimes according to (see Fig. 3)

$$f(x) = \frac{x}{(1 + x^{1.3})^{1/1.3}} \text{ .} \tag{22}$$

The exponent of 1.3 has been optimized to obtain a reasonable representation of the numerical data in the transition regime.

The second parameter for which an empirical formulation is needed is β. The physical meaning of β is clear from ~~Eq.~~ equation (8): it is the temperature of the top snow layer at the new time level T_1^{n+1} in case of zero heat flux. A simple

approximation would be to select the temperature of the previous time level, but this is only valid for a uniform temperature profile. For a non-uniform temperature profile, heat diffusion will homogenize temperature, which will make β different from T_1^n at the old time level. Following the scaling arguments above, we know that information propagates vertically over a distance δ during time step Δ. Therefore, we conjecture that the temperature of the old profile at depth δ is a better approximation for β

5   than the temperature at level 1, i.e. T_δ^n is better than T_1^n. Fig. 4 indeed confirms that the temperature at depth δ is a reasonable approximation. The temperature at $z = -\delta$ has been obtained by linear interpolation between levels, except when $\delta < 0.5\Delta z$. In the latter case, temperature T_1^n is selected. Note that, unlike α, β does change with temperature and does evolve during the integration.

From Figs. 3 and 4, it is concluded that reasonable estimates can be made of α and β without actually solving the tri-diagonal

10   matrix. Depth scale δ and the thickness of the top layer Δz are crucial scales to characterize the temperature evolution of the top snow layer over a time step.

6   Simulations with the empirical formulation

With the empirical formulations for α and β, it is possible now to repeat the simulations of section 4. Instead of generating the fully implicit solution by solving the tri-diagonal matrix in the standard way, α and β are replaced by the empirical formulation

15   between the elimination and back-substitution phase. If the formulation is perfect, the solution should be the same as the fully implicit solution. Results are shown in Fig. ~~?? for the temperature of the top layer~~ 5 for the skin temperature and the heat flux. Layer thicknesses of 0.2, 0.02 and 0.002 ~~m~~ m are shown as different rows in Fig. ~~??. In this case, the top layer temperature is shown instead of the skin temperature. The implication is that the amplitude of the diurnal cycle increases with the refinement of the vertical discretization, simply because with high vertical resolution, the top layer becomes a better~~

20   ~~approximation of the skin temperature. The~~ 5. The figure confirms that the diurnal temperature cycle of the fully implicit solution (blue curve, IMPL) is well reproduced by the solution with ~~parameterized~~ parametrized α and β (black solid cure, IMPPAR). The differences between blue and black curves are very small.

Finally, the scheme was further simplified by using the parametric form for α only and estimating β by putting it equal to T_1^n. The advantage is that no interpolation to $z = -\delta$ is needed, but that stability of the coupling is still maintained. However,

25   it is clear that numerical errors are increased for thin snow layers (see dashed black curve). Such errors have to be seen in the context of other model errors, so the use of a ~~parameterized~~ parametrized α only, to ensure stability, may still be sufficient for many applications.

7   Discussion and conclusion

~~For absolute numerical stability it is necessary to have a~~ Numerical stability is a critical issue for atmospheric models that are

30   coupled to a fast responding surface e.g. through a thin snow or ice layer. Such thin snow layers can occur in early winter after the first snow fall and during melt in spring. A fine discretization may also be desirable to allow for a fast response of

the surface temperature to changes in radiation. It is shown that stability can be achieved by fully implicit coupling of the heat diffusion between atmosphere and surface(e. g. snow). It.

Fully implicit coupling leads to a tri-diagonal problem in which atmosphere and surface are solved simultaneously. In practice, often so-called explicit flux coupling is applied: the atmospheric model uses the surface temperature of the previous time level to compute the surface heat flux, which is used later as boundary condition for the heat diffusion in the surface. Explicit surface coupling puts stability limits on the thickness of the top snow layer and on the time step. Explicit flux coupling is also desirable from the code modularity point of view.

Although the atmosphere / surface heat diffusion leads to a single tri-diagonal matrix problem, one can also break it up in different steps. It is shown that the elimination part of the solver of the snow heat diffusion problem leads to a linear relation between surface temperature and surface heat flux. This relation can be used together with the atmosphere / surface interaction formulation to solve for the surface heat flux.

A simple method has been developed to approximate the coefficients in this linear relation. The coefficients are scaled with the characteristic scales of the diffusion equation. This makes the result universal and applicable to an arbitrary medium e.g. snow, ice or soil. The depth scale that characterizes the penetration of a perturbation over a time step, turns out to play a crucial role. In this paper the relevant empirical function is "measured" by solving the diffusion equation for a range of vertical resolutions and time steps.

Finally, the empirical functions are used to solve for the coupled diffusion problem and compared with the fully implicit computations. The results are very close. The advantage of the method is that the surface fluxes can be computed without calling any surface code, and behaves like explicit flux coupling. The only difference is that the surface heat flux expression has a damping term depending on the time step. This damping term is the result of the change of surface temperature related to the heat flux, and stabilizes the result.

The scaling argument used above, only applies for a diffusion equation with constant properties of the medium. However, in reality there may a profile of e.g. snow density as snow becomes more and more compact in deeper layers, or vertical resolution may be variable. The latter is numerically equivalent to a variable diffusion coefficient [1]. As a simple test, a case was selected where the profile of density is 150 kgm^{-3} at the surface, increases linearly to 250 kgm^{-3} at a depth of 0.5 mm, and remains constant below 0.5 mm. The characteristic depth is again computed as in section 5, and to non-dimensionalize, the snow properties are taken from the middle of the top snow layer. For this case the dimensionless α and characteristic temperature β are shown in Figs. 6 and 7. They are very close to the figures for constant snow properties (Figs. 3 and 4), which suggests that the sensitivity to snow properties is fairly small. In general, it is to be expected that the snow properties very close to the surface control the relation between flux and temperature over a short time step, because the penetration depth δ is small.

We conclude that making an estimate of the relation between heat flux and surface temperature is a practical solution to support explicit flux coupling and to combine numerical stability for long time steps with a modular code structure. A formal stability analysis in Appendix B confirms unconditional stability of the proposed coupling method. The similarity framework

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[1] In fact the aerodynamic coupling between atmosphere and snow can be interpreted as a big jump in the properties of the medium

makes the method applicable to any medium, e.g. snow, ice or soil. It is also worth noting that the method does not compromise conservation: the heat flux that is computed by the atmospheric model is later used by the surface model as boundary condition.

Appendix A: Solving the tri-diagonal matrix equations

The set of equations discussed in section 2 leads to the following tri-diagonal system

$$
\begin{pmatrix}
B_1 & C_1 & 0 & 0 & \cdots & 0 \\
A_2 & B_2 & C_2 & 0 & \cdots & 0 \\
0 & A_3 & B_3 & C_3 & \cdots & 0 \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & A_{NL-2} & B_{NL-2} & C_{NL-2} & 0 \\
0 & \cdots & 0 & A_{NL-1} & B_{NL-1} & C_{NL-1} \\
0 & \cdots & 0 & 0 & A_{NL} & B_{NL}
\end{pmatrix}
\begin{pmatrix}
T_1^{n+1} \\
T_2^{n+1} \\
T_3^{n+1} \\
\vdots \\
T_{NL-2}^{n+1} \\
T_{NL-1}^{n+1} \\
T_{NL}^{n+1}
\end{pmatrix}
=
\begin{pmatrix}
R_1 \\
R_2 \\
R_3 \\
\vdots \\
R_{NL-2} \\
R_{NL-1} \\
R_{NL}
\end{pmatrix}
\tag{A1}
$$

where

$$
\begin{aligned}
A_j &= -\frac{K_{j-1/2}}{\Delta z_j\,\Delta z_{j-1/2}}\,, \\
B_j &= \frac{(\rho C)_j}{\Delta t} + \frac{K_{j-1/2}}{\Delta z_j\,\Delta z_{j-1/2}} + \frac{K_{j+1/2}}{\Delta z_j\,\Delta z_{j+1/2}}\,, \\
C_j &= -\frac{K_{j+1/2}}{\Delta z_j\,\Delta z_{j+1/2}}\,, \\
R_j &= \frac{(\rho C)j}{\Delta t}T_j^n\,,
\end{aligned}
\tag{A2}
$$

with boundary condition at the surface

$$
\begin{aligned}
A_1 &= 0\,, \\
B_1 &= \frac{(\rho C)_1}{\Delta t} + \frac{K_{1+1/2}}{\Delta z_1\,\Delta z_{1+1/2}}\,, \\
C_1 &= -\frac{K_{1+1/2}}{\Delta z_1\,\Delta z_{1+1/2}}\,, \\
R_1 &= \frac{G_0}{\Delta z_1} + \frac{(\rho C)_1}{\Delta t}T_1^n\,,
\end{aligned}
\tag{A3}
$$

and the no-flux condition at the bottom

$$
\begin{aligned}
A_{NL} &= -\frac{K_{NL-1/2}}{\Delta z_{NL}\,\Delta z_{NL-1/2}}\,, \\
B_{NL} &= \frac{(\rho C)_{NL}}{\Delta t} + \frac{K_{NL-1/2}}{\Delta z_{NL}\,\Delta z_{NL-1/2}}\,, \\
C_{NL} &= 0\,, \\
R_{NL} &= \frac{(\rho C)_{NL}}{\Delta t}T_{NL}^n\,.
\end{aligned}
\tag{A4}
$$

The tridiagonal system is solved in two steps by standard Gaussian elimination. The first step is an upward sweep to eliminate the C coefficients. It starts at level NL by rescaling coefficient A_{NL} to 1. The new coefficients identified by superscript * are

$$A_{NL}^* = 1\,, \quad B_{NL}^* = \frac{B_{NL}}{A_{NL}}\,, \quad R_{NL}^* = \frac{R_{NL}}{A_{NL}}\,. \tag{A5}$$

Next, coefficient C_{NL-1} is eliminated by multiplying A, B, C and R for level $NL-1$ by B_{NL}^*, the new coefficients for level NL with C_{NL-1} (i.e. from A5), subtracting the two equations and rescaling the result to have 1 at the position of A_{NL-1}. The new A, B, C and R for level $NL-1$ are

$$A_{NL-1}^* = 1\,, \quad B_{NL-1}^* = \frac{B_{NL-1}B_{NL}^* - C_{NL-1}}{A_{NL-1}B_{NL}^*}\,, \quad C_{NL-1}^* = 0\,, \quad R_{NL-1}^* = \frac{R_{NL-1}B_{NL}^* - C_{NL-1}R_{NL}^*}{A_{NL-1}B_{NL}^*}\,. \tag{A6}$$

This process is repeated to the top, which results in a matrix where the A-diagonals are all 1 and the C-diagonal contains zeros. If the surface heat flux is specified, the top line of the matrix contains the solution for T_1^{n+1}. The temperatures of all the other levels can be computed in a downward sweep, where the temperature of level j is used to find the solution for level $j+1$ with the equation for level $j+1$.

If the surface heat flux is not known, the first line of the matrix equation contains a linear relation between T_1^{n-1} and G_0, which can be written in the form of equation (8).

Appendix B: Stability analysis of the coupling schemes

In this section we present the stability properties of the three coupling methods introduced in the present study, namely the explicit flux coupling (EXPFLX), the implicit flux coupling (IMPFLX) and the so-called parametrized implicit flux coupling (IMPPAR). Since the numerical stability is expected to be greatly influenced by the numerical treatment of the surface boundary condition, a classical Von Neumann stability analysis, which assumes periodic boundary conditions, would not be adequate. For this reason our study is based on a matrix stability analysis (e.g. Oishi et al., 2008) . Assuming a constant grid spacing and diffusion coefficients, the results are shown in terms of the dimensionless coefficients σ and γ defined as

$$\sigma = \left(\frac{\delta}{\Delta z}\right)^2 = \frac{K\Delta t}{(\rho C)\Delta z^2} \qquad \gamma = \frac{\lambda_t \Delta t}{(\rho C)\Delta z}, \tag{B1}$$

where λ_t and δ are respectively defined in (11) and (17), with $t = \Delta t$. The typical value of those parameters for the numerical simulations discussed in sections 4 and 6 are given in table 2.

B1   Numerical treatment of the surface boundary condition

Without loss of generality we consider in the following that $T_a = 0$ in the computation of the surface boundary condition G_0 defined in (11) as well as that $K, \Delta z$, and ρC are held constant. The only difference between the three coupling algorithms considered here is in the treatment of the surface boundary condition (i.e. for the vertical index $j = 1$):

- *Explicit flux coupling:* the surface temperature involved in the computation of G_0 is T_1^n, thus leading to the following counterpart of equation (6)

$$(1+\sigma)\,T_1^{n+1} - \sigma T_2^{n+1} = (1-\gamma)T_1^n \tag{B2}$$

- *Implicit flux coupling:* the surface temperature at time level $n+1$ is used to compute G_0

$$(1+\sigma+\gamma)\,T_1^{n+1} - \sigma T_2^{n+1} = T_1^n \ . \tag{B3}$$

- *Parametrized implicit flux coupling*: the temperature at time level $n+1$ is diagnosed as $T_1^n/(1+\alpha\lambda_t)$

$$(1+\sigma)\,T_1^{n+1} - \sigma T_2^{n+1} = \left(1 - \frac{\gamma}{1+\alpha\lambda_t}\right) T_1^n \ . \tag{B4}$$

where α is defined in (21). Using (22) it can readily be seen that

$$\frac{\gamma}{1+\alpha\lambda_t} = \frac{\gamma}{1+\gamma(1+\sqrt{\sigma}^{1.3})^{-1/1.3}} \ , \tag{B5}$$

which shows that the parametrized implicit flux coupling can be interpreted as a limiter acting on the value of γ of the explicit flux coupling, indeed $\gamma(1+\alpha\lambda_t)^{-1} \le \gamma$.

B2 Matrix stability analysis

As shown in Appendix A, the Euler implicit scheme applied to the diffusion equation can be written in a general matrix form

$$\mathbf{A}\mathbf{T}^{n+1} = \mathbf{B}\mathbf{T}^n, \qquad \mathbf{T} = (T_1,\dots,T_{NL})^t, \tag{B6}$$

with

$$\mathbf{A} = \begin{pmatrix} 1+\sigma+\theta(2\theta-1)\gamma & -\sigma & 0 & 0 & \dots \\ -\sigma & 1+2\sigma & -\sigma & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \dots & 0 & -\sigma & 1+2\sigma & -\sigma \\ \dots & 0 & 0 & -\sigma & 1+\sigma \end{pmatrix}, \tag{B7}$$

$$\mathbf{B} = \begin{pmatrix} 1+(\theta-1)(1+2\frac{1-\alpha\lambda_t}{1+\alpha\lambda_t}\theta)\gamma & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & & \ddots & \ddots & \ddots & 0 \\ \dots & & 0 & 0 & 1 & 0 \\ \dots & & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{B8}$$

12

where $\theta = 0$ corresponds to the explicit flux coupling, $\theta = 1$ to the implicit flux coupling, and $\theta = 1/2$ to the parameterized implicit flux coupling. The general implicit scheme (B6) is stable if all the eigenvalues of the matrix $\mathbf{M} = \mathbf{A}^{-1}\mathbf{B}$ do not exceed 1 in magnitude. Therefore, the stability analysis requires the computation of the spectral radius of matrix $\mathbf{M}$, i.e. its larger eigenvalue in magnitude. For $\gamma \geq 0$ it can be shown that the smallest eigenvalue of $\mathbf{A}$ is larger or equal[2] to 1 meaning that this matrix is invertible for $\theta \in \{0, 1/2, 1\}$. In Fig. 8, values of the spectral radius of $\mathbf{M}$ obtained over a range of values of γ and σ are shown for each coupling algorithm[3]. Gray shaded areas coincide with regions where the spectral radius is larger than 1 thus indicating parameter values for which the corresponding scheme is unstable. From those results, the only algorithm that turns out to be conditionally stable is the explicit flux coupling whereas the implicit and parameterized implicit flux coupling are unconditionally stable. The results are thus consistent with the numerical experiments discussed in sections 4 and 6.

Empirically, it can be found that the stability condition for the explicit flux coupling roughly behaves like $\gamma \leq 2 + \sqrt{\sigma}^{1.1}$ (see figure 9). For the parametrized implicit flux coupling, γ is replaced by $\tilde{\gamma}(\sigma) = \frac{\gamma}{1 + \gamma(1 + \sqrt{\sigma}^{1.3})^{-1/1.3}}$ which is always smaller than $\tilde{\gamma}_{\max} = (1 + \sqrt{\sigma}^{1.3})^{1/1.3}$. As shown in figure 9, $\forall \sigma \geq 0$, $\tilde{\gamma}_{\max} \leq 2 + \sqrt{\sigma}^{1.1}$ meaning that for the particular choice of $f(\sqrt{\sigma})$ given in (22) the parameterized implicit flux coupling is unconditionally stable because it always satisfies the stability constraint of the explicit flux coupling.

Data availability

The data that is used in this paper has been produced with a dedicated stand-alone fortran program. ECMWF's data policy does not allow open access to software. However, the code can be obtained from the first author subject to license. The license implies non-commercial use i.e. for research and education only.

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[2] for the special cases $\theta = 0$ and $\theta = 1/2$, the eigenvalues $\lambda_k^{\mathbf{A}}$ of matrix $\mathbf{A}$ are given by $\lambda_k^{\mathbf{A}} = 1 + 2\sigma \left(1 + \cos \frac{k\pi}{NL}\right)$ for $k = 1, \ldots, NL$; therefore $\lambda_{\min}^{\mathbf{A}} = 1, \forall \sigma \geq 0$.

[3] Numerical results are obtained for $NL = 50$ after checking that an increased number of vertical levels does not change the results significantly. The use of very few vertical levels ($N \leq 10$) could lead to different stability results because only very few eigenmodes will be properly resolved. However, in this case we do not expect major stability issues since the values of σ and γ are very small, see table 2.
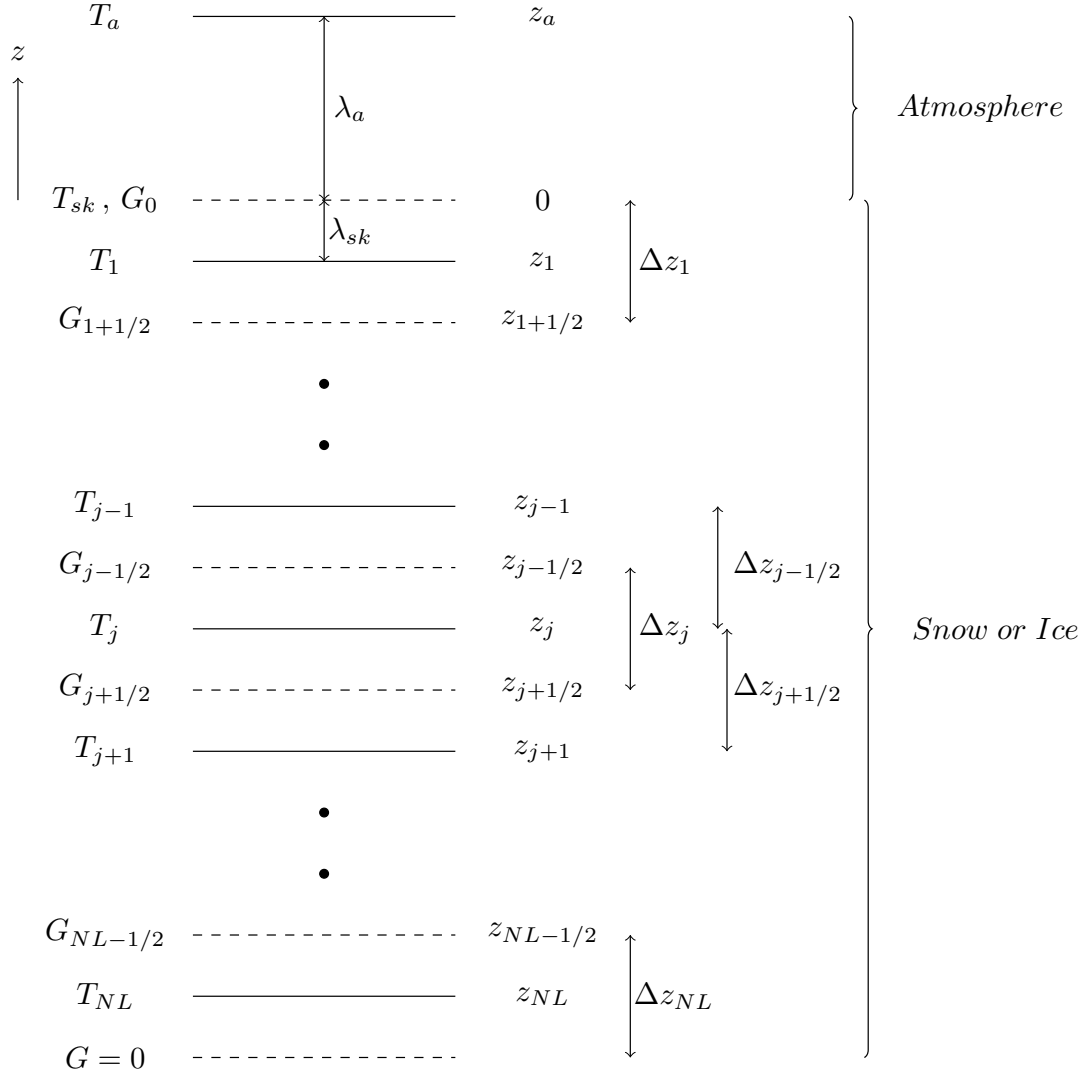
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**Table 1.** List of parameters used in the idealized simulation of a snow layer

Parameter	Description	Value	Units
ρ	snow density	150	$kg\,m^{-3}$
ρ_{ice}	ice density	920	$kg\,m^{-3}$
C	snow (and ice) heat capacity	2228	$J\,kg^{-1}K^{-1}$
K_{ice}	ice heat diffusion coefficient	2.2	$W\,m^{-1}K^{-1}$
K	snow heat diffusion coefficient	$K_{ice}(\rho/\rho_{ice})^{1.88}$	$W\,m^{-1}K^{-1}$
ρ_a	air density	1.2	$kg\,m^{-3}$
c_p	air heat capacity	1005	$J\,kg^{-1}K^{-1}$
$|U|$	absolute wind speed	4	$m\,s^{-1}$
z_{om}	roughness length for momentum	0.0001	m
z_{oh}	roughness length for heat	0.0001	m
z_a	height atmospheric forcing level	10	m
κ	VonKarman constant	0.4	$-$
D	total depth of snow layer	1	m

Table 2. Values of γ and σ defined in (B1) for various time steps Δt and vertical discretizations Δz, for $\rho = 150\,kg\,m^{-3}$, $C = 2228\,J\,kg^{-1}K^{-1}$, and $K = K_{ice}(\rho/\rho_{ice})^{1.88} \approx 7.27 \times 10^{-2}\,W\,m^{-1}K^{-1}$.

Δt	$[s]$	100	3600	100	3600	100	3600
Δz	$[m]$	0.2	0.2	0.02	0.02	0.002	0.002
λ_t	$[W\,m^{-2}K^{-1}]$	0.65	0.65	3.23	3.23	5.39	5.39
γ	$[-]$	9.6×10^{-4}	3.48×10^{-2}	4.81×10^{-2}	1.74	0.8	29
σ	$[-]$	5.44×10^{-4}	1.96×10^{-2}	5.44×10^{-2}	1.96	5.44	195.8

T_a z_a

z

λ_a *Atmosphere*

T_{sk} , G_0 0

λ_{sk}

T_1 z_1 Δz_1

$G_{1+1/2}$ $z_{1+1/2}$

T_{j-1} z_{j-1}

$G_{j-1/2}$ $z_{j-1/2}$ $\Delta z_{j-1/2}$

T_j z_j Δz_j *Snow or Ice*

$G_{j+1/2}$ $z_{j+1/2}$ $\Delta z_{j+1/2}$

T_{j+1} z_{j+1}

$G_{NL-1/2}$ $z_{NL-1/2}$

T_{NL} z_{NL} Δz_{NL}

$G = 0$

Figure 1. The numerical grid is defined by the position of the half levels, i.e. the thickness of the layers. The full levels are in the middle of the layers, i.e. $z_j = (z_{j-1/2} + z_{j+1/2})/2$. The surface is at $z = 0$. The bottom level is defined by the accumulated depth of all the layers. The temperature is defined on full levels and the heat fluxes are defined on half levels.
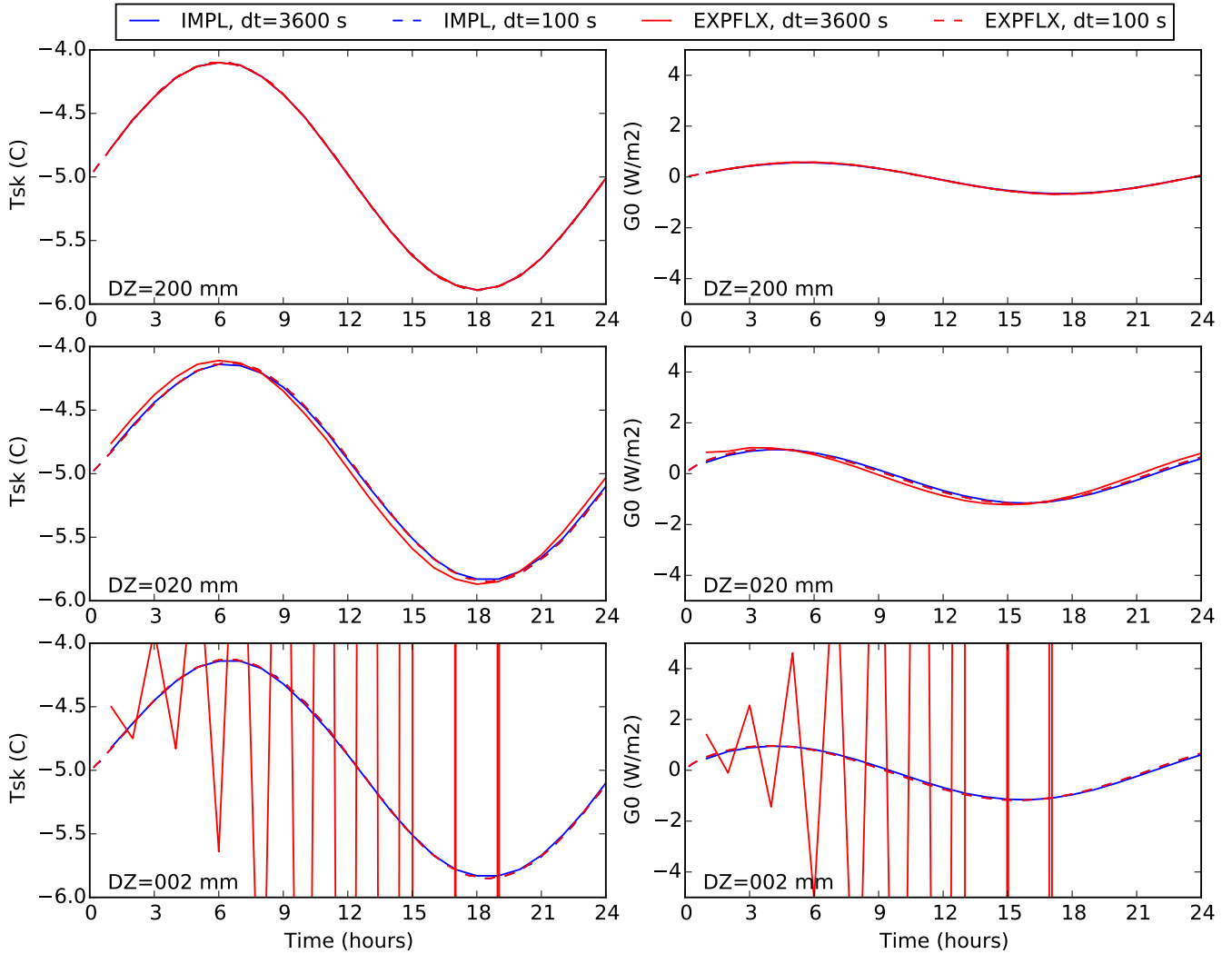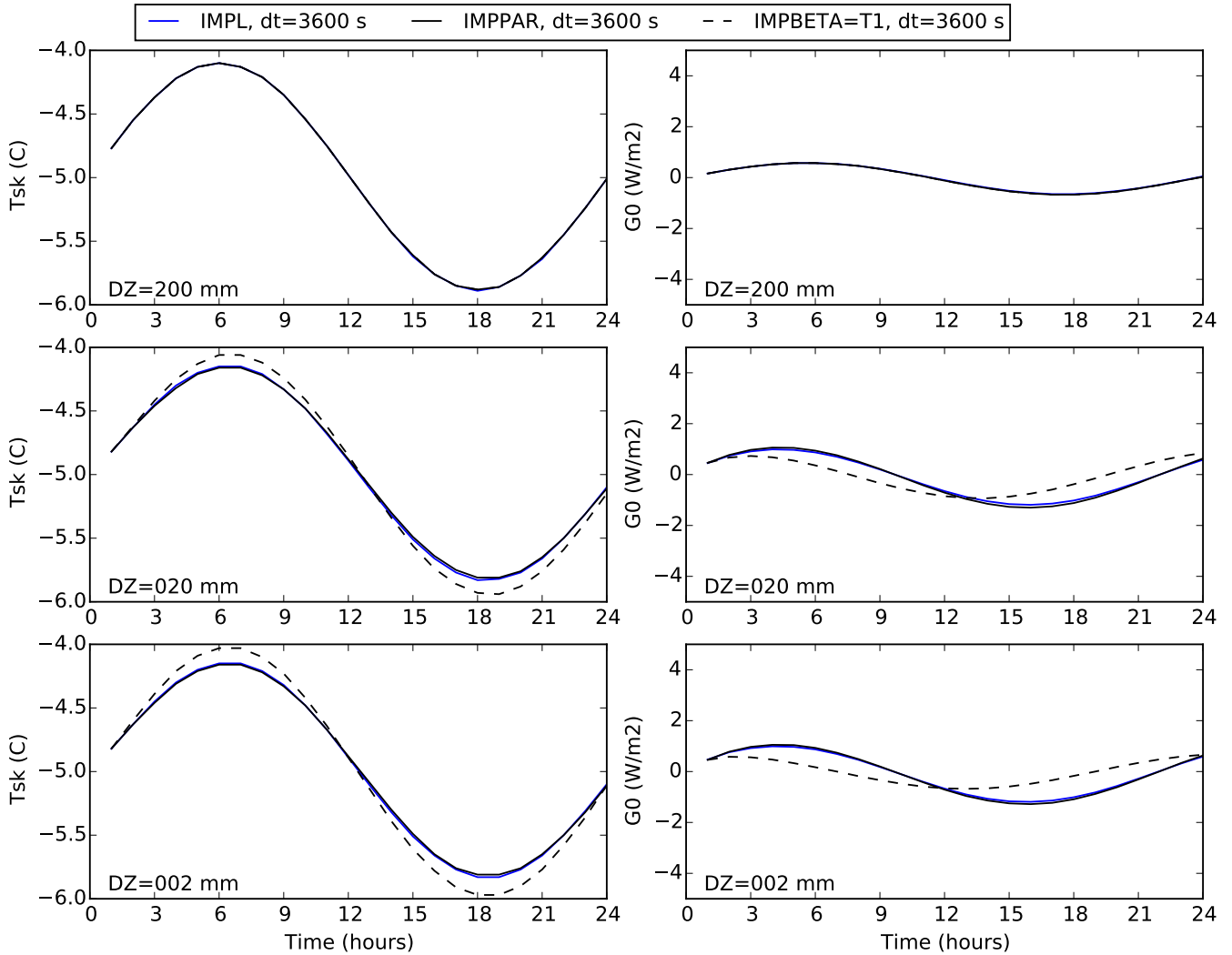
Figure 2. Diurnal cycle time series of snow skin temperature (left column) and surface heat flux (right column). The simulations were made with 0.2, 0.02 and 0.002 ~~m~~ *m* vertical resolution (top, middle and bottom panels). The blue curves refer to the fully implicit solution(IMPL); the red curves indicate the solutions with explicit flux coupling (EXPFLX). The solid curves are with a time step of 3600 *s* seconds and the dashed curves with 100 seconds.
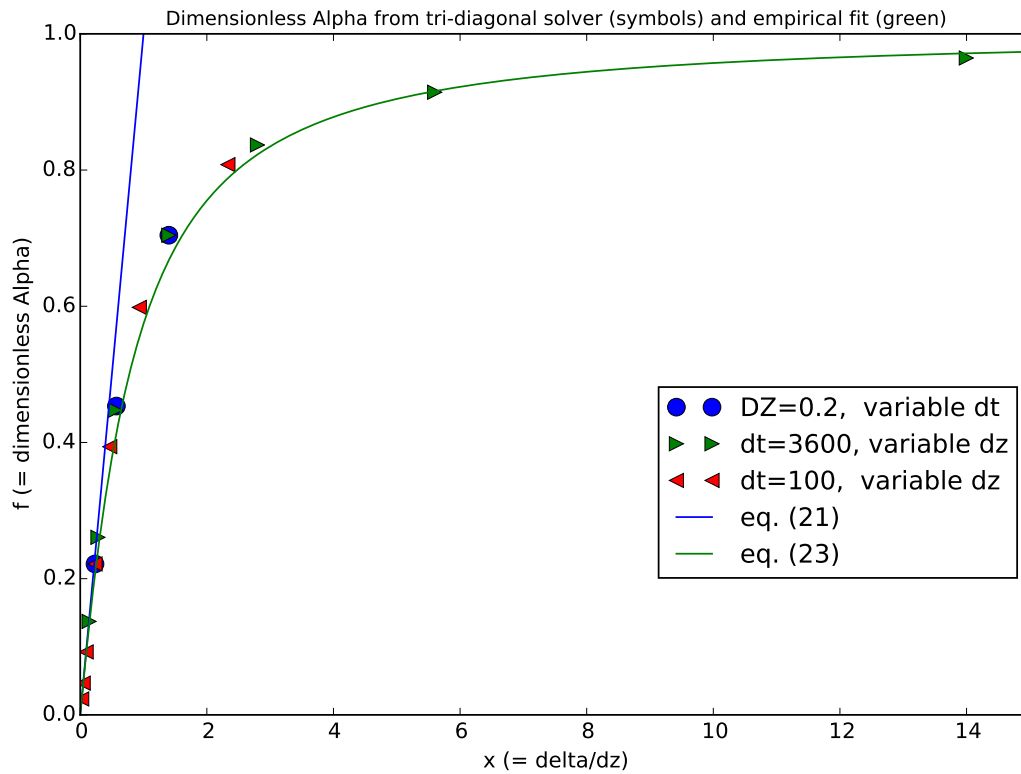
Figure 3. Dimensionless function $f = \alpha(K\,\rho C/\Delta t)^{1/2}$ as a function of $x = \delta/\Delta z$. The circles and triangles are for different combinations of Δz and Δt. The blue line is the asymptotic limit for small $\delta/\Delta z$. The green curve is the empirical fit according to ~~Eq.~~ equation (22).
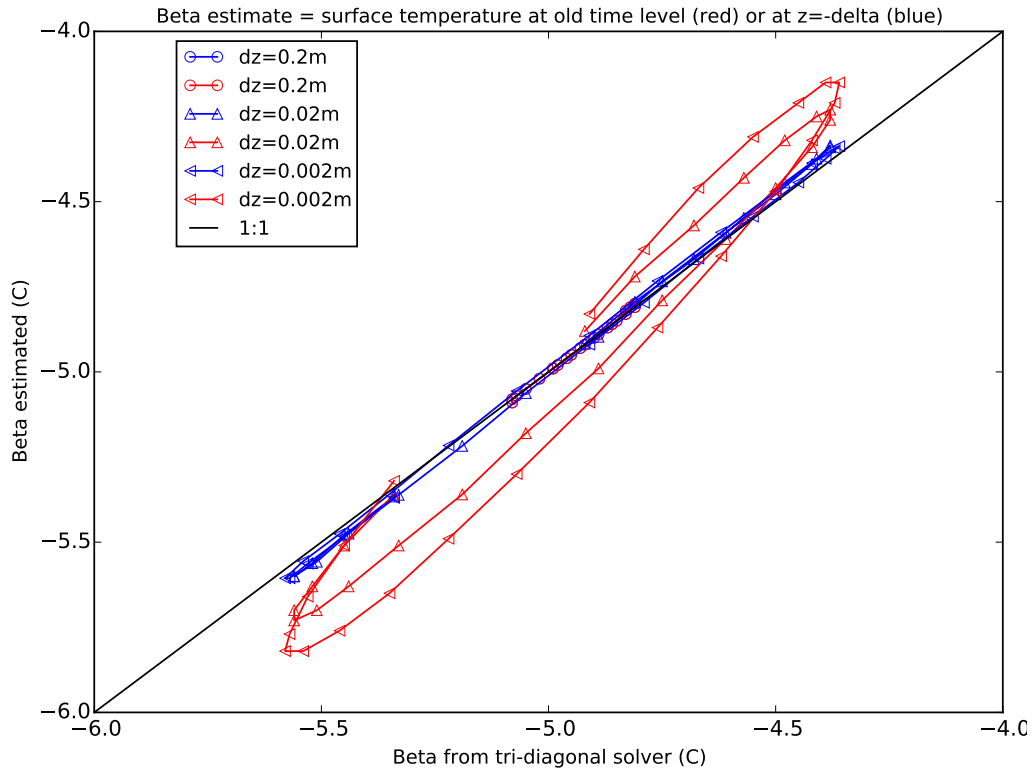
Figure 4. Empirical estimates of parameter β as a function of the value found from the tri-diagonal solver. The red curve represents the estimate according to T_1^n and the blue curve is the temperature at $z = -\delta$, also at the previous time level n. The symbols (connected by lines) indicate the successive time steps in the diurnal cycle. Results are plotted for vertical resolutions of 0.2, 0.02 and 0.002 ~~m~~ m.

Figure 5. Diurnal cycle series of ~~top layer snow~~ skin temperature (left columns and) and surface heat flux (right columns). The simulations were made with 0.2, 0.02 and 0.002 ~~m~~ m resolution (top, middle and bottom panels). The blue curve refers to the fully implicit solution(IMPL); the black solid curve is the solution with ~~parameterized~~ parametrized α and β. The black dashes curve refers to the solution where α is ~~parameterized~~ parametrized and β is set equal to the temperature of level 1 at the previous time (n). The time step is 3600 seconds.
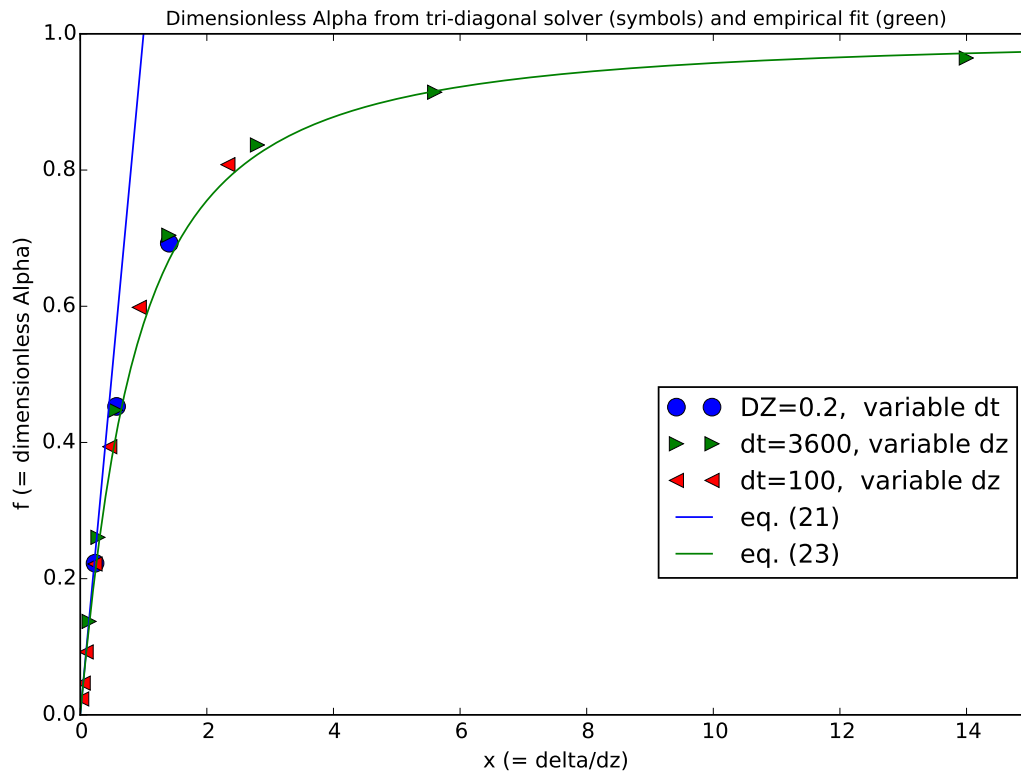
Figure 6. Dimensionless α as in Fig. 3, but for non-uniform snow density. The snow density is 150 kgm^{-3} at the surface, increases linearly to 250 kgm^{-3} at a depth of 0.5 ~~mm~~m, and remains constant below 0.5 ~~mm~~m.
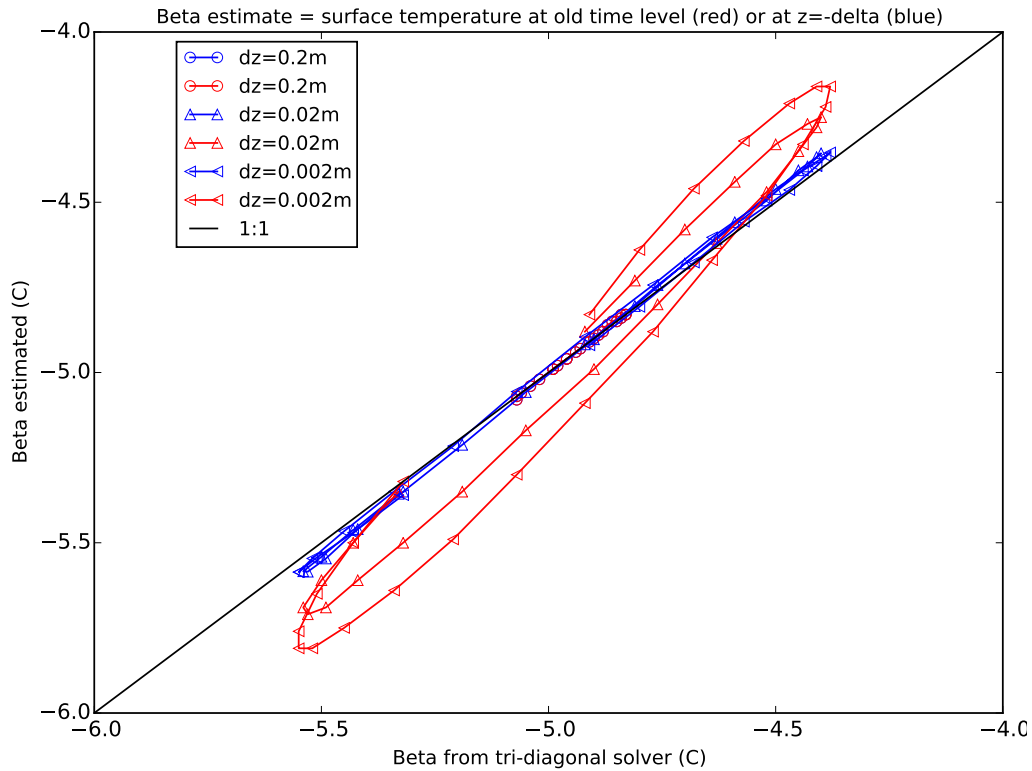
Figure 7. Dimensionless β as in Fig. 4, but for non-uniform snow density. The snow density is 150 kgm^{-3} at the surface, increases linearly to 250 kgm^{-3} at a depth of 0.5 ~~mm~~m, and remains constant below 0.5 ~~mm~~m.
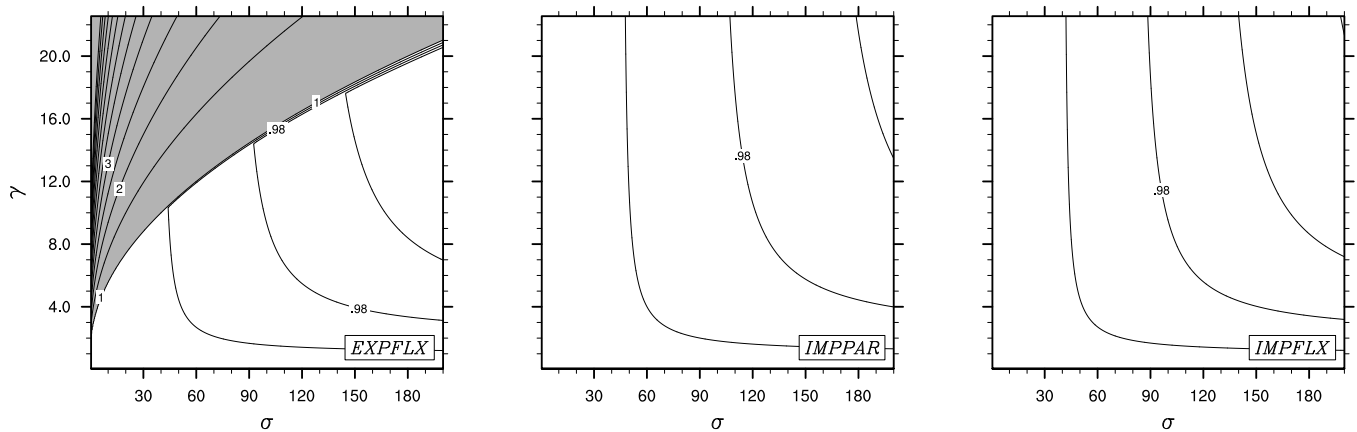
Figure 8. Spectral radius of the matrix $\mathbf{M} = \mathbf{A}^{-1}\mathbf{B}$ (defined in B7,B8) associated to the explicit flux coupling (EXPFLX), parametrized implicit flux coupling (IMPPAR), and implicit flux coupling (IMPFLX) with respect to the dimensionless coefficients γ and σ. Gray shaded areas correspond to regions where the spectral radius is larger than 1.
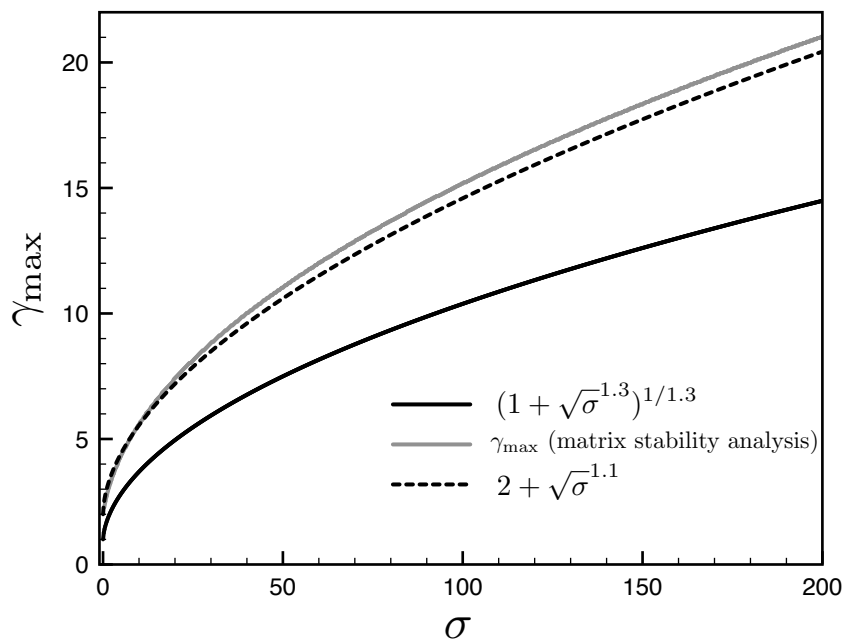
Figure 9. Maximum value of γ with respect to the parabolic Courant number σ to guarantee stability of the explicit flux coupling (solid gray) which roughly behaves like $\gamma(\sigma) = 2 + \sqrt{\sigma}^{1.1}$ (doted black line). The parametrized implicit flux coupling replaces γ by $\tilde{\gamma}$ which is always smaller than $\tilde{\gamma}_{max} = (1 + \sqrt{\sigma}^{1.3})^{1/1.3}$ (solid black line). Since the solid black line is below the solid gray line, the implicit flux parametrization is unconditionally stable.