Answer to referee #1

General Comments : For this paper to warrant publication, it would require its methods to be outlined in much greater detail than they currently are. The primary nuance of the work, mainly the computation of R from a surface DEM is given only a few lines of description. The metrics by which the authors judge quality are primarily left unstated. Many modelling assumption (i.e. that surface elevations can act as a reasonable proxy for the direction of a Stokes'-based velocity field) are utilized but unjustifed. Suffice to say, this work would not be repeatable based upon the information given here. I would like to offer up an opinion on whether the results presented are significant to the field of glaciology, but lack a sufficient understanding of the quality of the results to do so.

Authors' answer : Several points needed to be specified to make the article more intelligible ; in particular, we added information about the assumptions of the model (emphasize the fact that it is done for divides or center lines of drainage basins), and our method to process to the twin experiment between 3D and 2.5D.

The paper is rife with incorrect grammar, spelling, and otherwise improper English, to the extent that it partially inhibits understanding of the methods and results contained herein. As a reviewer, I don't really see it as my responsibility to correct for these issues, particularly when at least one of the co-authors is a native English speaker, and I will not provide comments on these points. As such, I would suggest that the authors employ or otherwise recruit a capable copy editor to polish the manuscript. It is presently not ready for publication for this reason, scientific merit aside.

Specific Comments : Abstract : the abstract introduces a significant amount of jargon, such as "scanning window", "reversed surface convexity", and "partly reversed velocity profile", which the reader cannot know the meaning of without reading the paper. This undercuts the purpose of the abstract.

Authors' answer : We explained the jargon, or removed it to make the sentences simpler. The english has been improved by a carreful re-reading from the english speaker of the team.

P1 L8 : The relief in question is on the order of tens of meters, yet the authors suggest that the so-called "scanning window" should be on the order of kilometers. This directly contradicts this line.

Authors' answer : "relief" was erroneous. The sentence was modified to mention the radius of the surface curvature.

P1 L16 : the equations in question are either Stokes' equations, or they are not ; the "full-" modifier is unnecessary. Authors' answer : Modification done

P1 L21 : Durand (2011) does not appear to actually address this issue, though it does use a flowline model. Also, what are the "regular flanks of an ice sheet" ?

Authors' answer : Reference to Durand (2011) removed. "Ice-sheet" was changed in "ice cap". In Martin et al, 2006, the Roosevelt Island shows a sharp axis of symmetry, we made the sentence more specific.

P1 L22 : "plane strain" does not seem to be used correctly here.

Authors' answer : Precision was added, but we don't agree and think that plane strain is the appropriate description of the state of strain we want to refer to.

P2 L1 : I think I know what is meant by "a particular vertical surface", but this would be greatly clarified by addition of said surface to Fig 1.

Authors' answer : Fig.1 has been modified, showing the intersection between a vertical plane and the ice surface.

P2 L14 : Why is the parenthetical "(column-flow model)" included here ? It does not seem to serve a purpose and doesn't seem to be referenced later.

Authors' answer : 1-Since the 2.5D model is the only object of the paper, it's worth remembering who developped this kind of model first, and who improved upon that. 2 - As we have some troubles with the velocity profiles computed with diverging geometries, column-flow models could still be useful (§4.3).

P2 L22 : "In particular,..." I do not understand this sentence.

Authors' answer : We modified the sentence.

P2 L31 : width of the flow tube ?

Authors' answer : "Tube" added

P2 L31 : A DEM gives the shape of the surface and its contour lines when velocity is available, as well.

Authors' answer : We changed the sentence, in the way that the DEM method is the only possible one when surface velocities are unknown.

P2 L34 : Assuming that flow is always oriented along the surface gradient is a wrong assumption, and the differences between considering gradient-based curvature and velocity-based width goes far beyond issues of numerical accuracy. Have a look at any computation of balance velocity ever made ; if flow is routed down-gradient then the surface velocity looks like a river network, which is silly. Longitudinal stresses matter for flow routing !

Authors' answer : The text is now more specific on the cases for which the model is built. In particular it should hold for flowlines along ice divides and center lines of drainage basins.

P2 L35 : I don't know what "scanned" means, and as such I cannot assess whether the authors' statements about ambiguity in surface curvature have any merit. It would be better to hold off on elaborating upon it further until some basic definitions have been stated.

Authors' answer : We now give a basic description of the DEM method here.

P3 L10 : On "diverging geometries" : flow fields can diverge, because they are vector fields. Scalars cannot diverge, and the fields representing geometry are certainly scalars.

Authors' answer : Changed for "diverging flows"

P3 L17 : What "1D parameter" are we talking about here ? Where is this question returned to in the rest of the text ?

Authors' answer : The 1D parameter is R(x), which is explicitely mentionned now. We added a more specific answer to the question in the conclusion.

P3 L25 : Please state more clearly that the 3D model is used to construct a set of surface geometries based on different choices of lateral boundary conditions. This 3D model output is then taken as input to the 2.5D model. In particular the computed surface elevations from the 3D model are used to generate the curvature for the 2.5D case.

Authors' answer : We added the suggested precisions.

P3 L27 : "Finally we discuss the importance of ...". I do not understand this sentence. "Certain authors" ? Meaning the authors of this paper ? Or Reeh ?

Authors' answer : Sentence deleted, since this particular experiment has been suggested to be removed.

P3 L31 : "Similar synthetic geometry". Similar to what ?

Authors' answer : "Similar" was erroneous

Sec. 2.1. : It would be useful to specify, as did Gillet-Chaulet and Hindmarsh (2011), that the edges of the domain do not correspond to a physical boundary. Indeed, the authors could draw a considerable amount of inspiration from that paper on how to describe the setup of the model experiments. To understand what was being done in this paper I had to read that paper, and most readers would appreciate eliminating the intermediate step.

Authors' answer : Additional explanations are provided, specifying the virtuality of the boundary and the way the BC is constructed.

Sec. 2.1. : Please distinguish between a dome (what is ostensibly being modeled) and a cylinder (the shape of the mesh).

Authors' answer : "cylinder" now replaces "dome"

Sec. 2.3.1 : If the authors are unwilling to state the Stokes' equations completely, then it might be best to not state them at all, and just give a reference to one of Elmer/Ice's numerous model description papers. Otherwise, there are many missing definitions (e.g. $A(T)$,ϵ, etc.).

Authors' answer : We now specify a reference concerning the temperature dependence of the fluidity. The strain rate component are defined at the begining of the subsection.

P4 L8 : The strain rate tensor (written in the paper as ϵ) needs a dot over it.

Authors' answer : dot added

P5 L8 : Calling a a function is confusing in this context, because it's a constant. Just call it the accumulation rate.

Authors' answer : "rate" replaces "function"

Eqs. 9/10 : Trigonometric functions are usually typeset upright, rather than in italics.

Authors' answer : upright done

P5 L20 : L is not used, so why is it defined here ?

Authors' answer : We now specify that x=L is the downstream position, just for the reader to understand why the subscript L appears.

P6 L2 : It would be useful to use different coordinate symbols for the global Cartesian system that the 3D model uses versus the local system used by the 2.5D model.

Authors' answer : Since we never need a curvilinear system in our case, defining additional notations could be confusing. We mention the curvilinear system because the model was primarily designed with these coordinates.

P6 L13 : Asserting that an assumption is reasonable requires a reference.

Authors' answer : Reference added (Reeh, 1988)

P6 Sec. 3.1 : This section needs some clarification. Is it the ridge which runs along y = 0 or the centerline of the 2.5D model coordinate system ?

Authors' answer : Specification is given. y=0 and x=0 are taken on the 3D dome, and the 2.5D is run along one of these direction.

P6 L24 : If a flow tube diverges, then the tube surface area gets larger, and velocities are reduced because an equal flux moves through a larger area.

Authors' answer : For a given output width, the more divergent the flow, the narrower the tube. For example : if there is no divergence, all the Fig. 1 would be gray, and for linear divergence (axisymmetry), half of it would be gray. For higher divergence, the tube would be even narrower.

P7 L20 : Why neglect transverse shear stresses (i.e. σ_{xy}) ? Elmer can solve Stokes' equations, so technically it shouldn't be that difficult to include them here. There are plenty of width parameterizations out there (see, the works of Vanderveen on fjord wall drag, for example).

Authors' answer : We wanted to use a 2.5D model because the geometry of our future area of interest in Antarctica well corresponds to the assumptions of Reeh (1988). That's why we are putting ourselves in its direct following. In this frame, including transverse stresses would not allow to get rid of the y-coordinate. Concerning this questioning, the additional explanations of the model of Reeh (1988) in the introduction can help as well.

P7 L20 : Please provide a page number for Jaeger (1969). Also, consider citing Hvidberg (1996) instead, as this published article is much more readily available than a PhD thesis. Also, according to Hvidberg (1996), this result is derived for the axisymmetric case. Can the authors show that it remains valid in the case where this assumption is violated (i.e. $\alpha > 2$). In any case, these equations need considerably more explanation.

Authors' answer : The model can be considered as a generalized axisymmetrical model (as you can see in Hvidberg et al., Ice flow between the Greenland Ice Core Project and Greenland Ice Sheet Project 2 boreholes in central Greenland, 1997). The paper of Hvidberg 1996 only deals with axisymmetry since she specifies that R=x, but she gives the general version of the eqations anyway. More information about the equations are now provided.

P8 L3 : Do the authors mean "imposed horizontal velocity profile" ?

Authors' answer : "Horizontal" now replaces "Vertical"

P8 L12 : I like Elmer too, but its efficiency isn't really relevant here.

Authors' answer : Removed

P8 Sec. 3.4 : This section needs to be expanded greatly, and could do with some illustrative figures. After reading this, I really still have no idea where R comes from, and why changing the size of the sample changes this. If the surface contours are well approximated by polynomials, I fail to see why a small window shouldn't perform equally to a large window. This is really the crux of the method and a discussion of it takes up a bulk of the results, yet it is given only a few sentences in the methods. How can a reader assess the validity of the results without knowing what the authors did ?

Authors' answer : We give more explanations, in particular concerning the difference between large and small windows. In fact, the question is what is the typical distance which influences the local ice flow, and how we could account for it.

P9 L3 : Calling the case with the closed-form R = x "2D" rather than 2.5D is confusing. The extra half-dimension comes from the fact that width variations are being parame- terized, and that's still the case when R is known exactly. Much better would be to call these runs "2.5D with analytic R" or something like that.

Authors' answer : We changed the formulation for "2.5D"

Sec. 4.1.2 : I wonder if using a different method to compute widths would be less error-prone. For example, since we're already assuming that the flow follows the surface gradient, why not try just finding two flowlines and computing the distance between them ? That eliminates the need to compute a second derivative (usually an error prone activity). I guess if the rationale behind the way that curvature is computed were more fully explained, the answer to this question might be obvious.

Authors' answer : Even if we determined the width of the flow tube by tracking particles, the equations are anyway parameterized with R. As given by the equation 12, we would need to compute the derivative of the width. Furthermore, the method to track particles would not be efficient to evaluate the divergence on the top of the ridge : it would be impossible to distinguish between the particles, since they would be inside the same pixel on part of the trajectory (and even 100 % of the trajectory if the tube is much diverging).

Sec. 4.1.2 : A bit of specificity beyond "an error range of 0-10%" is in order here. How about just reporting standard error ?

Authors' answer : We now indicate the root mean square error (standard error could lead to compensation between negative and positive terms, and the final value could be deceptive.)

Fig. 4,5,6,8 : All need a legend.

Authors' answer : The legends were added.

P10 L7 : I don't understand what this paragraph is saying, nor do I find any clues in Fig. 6. What is the significance of a concave surface slope ?

Authors' answer : We describe more specifically the shape of the surface. The problem that appears is numerical, and comes from the balance between the terms of the mass conservation equation when close to the singularity.

P10 L19 : "the error made with this more complete model seems now to be small enough to be used for dating purposes." What evidence presented herein supports this point ? To make such a claim, the authors need to establish an error threshold (a priori) which must be met in order to claim that their method is accurate, and then go about showing quantitatively that the model performs up to this standard. At no point do I see any objective metrics for model performance in this regard.

Authors' answer : The text now mention the tracking of the ice particles, which is linked to the velocity field (so it is the same idea, but not mentionning a totally different subject).

P10 L25 : What is the error here, and in what way is it consistent with Hvidberg (1997b) ? Am I to take away that the 2.5D model does a better job perpendicular to the dome ?

Authors' answer : We give the RMSE for these cases. The 2.5D model meets difficulties to handle high divergence, so it is logical that the results are better perpendicular to the sharpest ridge. The reference to Hvidberg et al. (1997b) was simply indicative, and the error in u is in fact of the same order of magnitude than the error in R. It would be quite heavy to properly show it here, so we removed the sentence.

P10 L31 : I am skeptical of the authors' hypothesis that there is significantly different flow directions at different points in the ice column, primarily because this would lead to symmetry breaking that does not occur in any models that I know of. It would be easy to test this idea, since evidently the authors have the full 3D model output in hand. I don't really know what's causing the strange non-physical vertical inversion of the horizontal velocity profile, but I suspect it has to do with the neglect of transverse shear stresses or vertical resistive stresses.

Authors' answer : The only parameter that changes between the two experiments is the temperature. If the transverse shear stress or vertical resistive stress were involved, they would occur for the isothermal case as well. The enlargement of the tube, consequence of the different orientation of the flow along the vertical, comes from the 3D model. It is possible that, this phenomenon being very particular and problematic only in the case of 2.5D model, it has received little attention.

P11 L7 : I am not familiar with the results from Hvidberg (2002). It would be helpful if the authors restated them.

Authors' answer : More specific informations on Hvidberg et al. (2002) have been added.

Sec 4.4 : This section is a non-sequitur. What is the "mass-only conservation model" ? Are the authors referring to a calculation of balance velocity ? If so, there are numerical considerations and different boundary conditions that the authors do not state. Unless the authors make a considerable effort to define what the model results that they are referring to actually are, this section should be removed.

Authors' answer : This section is removed.

Conclusions : This sections seems to be an afterthought ; it is too short, ambiguous, and the conclusions stated herein are not clearly supported by the text.

Authors' answer : Two additional paragraphs have been added that steps back and takes stock.

Appendix : with respect to typesetting, the dot used to indicate scalar multiplication is not necessary.

Authors' answer : dot deleted

Answer to referee #2

This manuscript presents a study the performance of a 2.5D model versus a full 3D model and its applicability in the vicinity of a dome. Ice flow is complex and boundary conditions are not easily parameterized and well constrained by observations. Ice flow is described by a set of thermo-mechanically coupled non-linear differential equations and the numerical solution of these equations is computationally very demanding. Simulations of ice flow are often done on simplified systems, and a commonly used approximation is to reduce the 3D set of equations to a 2.5 flowline version. Investigating the applicability and performance of these models is therefore an interesting contribution to the community.

It is important to note that this test has not been done before. One reason is that only recently complex models that solves the full set of stress balance equations have become available to do the test. These models are still so computationally demanding, however, that simplifications of the equations are required for many purposes.

The study is focused and well-structured. The model is presented, both the continuum mechanically based set of equations and boundary conditions as well as the numerical implementation. The set of equations are presented without further references and arguments for the choices of model parameters. It is clear that the model is run for Antarctic conditions (temperature conditions), but this is not mentioned. A little information on the choices of model parameters and possible effects would clarify (from the simplified temperature, and the chosen temperature regime).

Authors' answer : We added some information about our global frame (our final goal is to work on a small dome in Antarctica), and we are more specific on the effects of temperature on viscosity.

More details on how R is determined from the DEM using a scanning window are needed, for example - how is the fit done, - explain that R is not constant within the window. . .. How R is determined is a critical parameter, for example the size of the scanning window, and the details of how this is done should be clearly described.

Authors' answer : More information is given concerning the determination of R during our twin experiment, especially what we expect from a small or a large window. In fact, the question is what is the typical distance which influences the local ice flow.

The effects when moving from 2.5D to 3D are complicated and result in surprising effects. It is surprising to see how the uncertainty of the radius of curvature for a small scanning window completely dominates in figure 5. It is very interesting to see the distribution of horizontal velocity fields for the non-isothermal case (figure 7). In 2.5D these variations may lead to spurious effects if used to model internal layers within the ice. A short paragraph should be included (introduction and/or conclusion) to mention this and thereby emphasize the significance of the results presented in the manuscript.

Authors' answer : The issue of modelling the internal layers, which is indeed a possible goal of using such a model, is now addressed in the conclusion.

The manuscript only considers 2.5D flow along straight lines. Sometimes 2.5D models are being used along curved flowlines, and neglecting the curvature of the flowline would add further to the uncertainties. It would be difficult to say something general about curving flow lines, so I do not suggest further studies, but the problem with curving flow lines should be mentioned in the manuscript.

Authors' answer : This issue is now mentionned in the conclusion as well.

I do not understand the comparison presented in section 4.4. The mass-only conservation model is not explained in detail, and does not add further to the conclusions. I am also suspicious about the boundary effects near x= 15000 m. They are not discussed but clearly influences the solution. I suggest that this section is removed.

Authors' answer : As both referees suggested to remove this section, we removed it.

There are several examples of incorrect use of English (e.g. order of words in a sentence), and I suggest that the manuscript is carefully worked through to clarify the text. The structure of the manuscript is well planned, and overall the manuscript appears clear and with a logical flow. The figures are clear and well presented.

To conclude, I find that the manuscript is relevant and provides a needed insights into the applicability of 2.5D models. The results can help clarify the performance and limitations of these models, which has not been systematically done before. The results also demonstrate that full stress solutions are needed near domes and divides to fully represent the flow. I recommend that the manuscript is published with minor changes mentioned above, as well as a thorough correction of the use of English in the text.

Relevant changes

The structure of the introduction was significantly rehandled to make the reasoning progress more logically.

Additional explanations were given in §3.2.2 and §3.4, concerning the momentum conservation, and the effect of the size of the window, respectively.

The conclusion was extented, to compare the sources of error in 2.5D models, and to adress some issues concerning the use of 2.5D model in real cases.

Performance and applicability of a 2.5D ice-flow model in the vicinity of a dome

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Abstract. Three-dimensional ice flow modelling requires a large amount of computing resources and observation data, such that 2D simulations are often preferable. However, when there is significant lateral divergence, this must be accounted for (2.5D models), and a flow tube is considered (volume between two horizontal flow lines). In the absence of velocity observations, this flow tube can be derived assuming that the flow lines follow the steepest slope of the surface, under a few flow assumptions. This method typically consists of scanning a digital elevation model (DEM) with a moving window, and computing the curvature at the center of this window. The ability of the 2.5D models to account properly for a 3D state of strain and stress has not clearly been established, nor their sensitivity to the size of the scanning window, and to the geometry of the ice surface, for example in the cases of sharp ridges. Here, we study the applicability of a 2.5D ice flow model around a dome, typical of the East Antarctic plateau conditions. A twin experiment is carried out, comparing 3D and 2.5D computed velocities, on three dome geometries, for several scanning windows and thermal conditions. The chosen scanning window used to evaluate the ice surface curvature should be comparable to the typical radius of this curvature. For isothermal ice, the error made by the 2.5D model is in the range 0-10% for weakly diverging flows, but is 2 or 3 times higher for highly diverging flows and could lead to a non-physical ice surface at the dome. For non-isothermal ice, assuming a linear temperature profile, the presence of a sharp ridge makes the 2.5D velocity field unrealistic. In such cases, the basal ice is warmer and more easily laterally strained than the upper one, the walls of the flow tube are not vertical, and the assumptions of the 2.5D model are no longer valid.

1 Introduction

Computing performance has continued to increase through the last decade, and 3D numerical simulations that could not be performed a few years ago are nowadays affordable. In particular, the Stokes equations can be solved on large 3D datasets (e.g., Gillet-Chaulet et al., 2012), such that the complete state of stress is accounted for. Nevertheless, two-dimensional (x,z) models are still very useful, since they are easier to handle, the computing time is at least two orders of magnitude less, and they need fewer observations to describe the boundary conditions. Two-dimensional models assume to be unchanging in the transverse direction, such that they apply well when the topography is always the same in the missing y-direction. This is a reasonable assumption for the case of large ice streams, or on the flanks of a symmetric ice-cap (e.g., Martín et al., 2006).

Under this assumption, the ice undergoes a strain in a vertical plane only, and all the components of strain in the transverse direction vanish (*plane strain* case).

In this study, we describe the volume between theoretical stream lines as a "flow tube". (Fig. 1, red lines). If the ice flow is locally converging or diverging, 2D models are no longer valid, as they do not conserve the mass in this case, and cannot account for lateral mechanical stresses imposed by the width variations of the flow tube. For example, Sergienko (2012) showed that the 2D (x, z) models could not account for the flow of ice on a bumpy bed, because they consider no lateral variability, and recommended to use such models on a width-averaged topography. It is however possible to account for variations in the width of the flow tube in the model, and still express the equations with (x, z) coordinates only (Reeh, 1988); we hereafter call these models 2.5D models. Under a few flow assumptions (the bed elevation varies slowly in the transverse direction, the walls of the flow tube are vertical, the lateral curvature of the flow line is small), such models represent an improvement on 2D models because they account for lateral variability, while maintaining the computational speed which full 3D models lack.

A particularly simple case is that of an axisymmetric circular dome, where the width of the flow tube increases linearly along the flowline. In any other case, there are several possible methods for determining the width of the flow tube. At a large scale, the ice velocity can be determined via interferometric synthetic-aperture radar data (Rignot et al., 2011), but this is only reliable when ice velocity is large compared to the associated error. In the interior of ice sheets, where the ice velocity is low, velocity measurements can be made using stake monitoring. If correctly positioned, the direction and position of the stakes may give the width of the flow tube (Waddington et al., 2007). Without any available velocity observations, the only possibility is to use a digital elevation model (DEM) describing the shape of the surface. The velocity measurement method is preferable when possible, since the ice does not always flow perpendicular to the surface contour lines. However, along a divide, or along the centre line of a drainage basin, the longitudinal surface velocity reaches its minimum or its maximum value compared to the lateral surroundings; moreover, if the flow line has a slight horizontal curvature, the horizontal shear strain rate can be neglected (Reeh, 1988). The ice velocity is then oriented along the steepest surface slope and the curvature of the surface contour lines linked to the width of the flow tube; for a dome or a two-dimensional ridge, determining the curvature of the surface contour line is sufficient to describe the widening of the flow tube. In this case, to determine the local curvature of a surface, the neighborhood of each point of a DEM is approximated via interpolation. The size of this neighborhood (scanning window) can be freely chosen, but no objective criteria are currently available to make this choice. This approach has been used to compute the surface curvature of an ice sheet (Rémy and Minster, 1997; Rémy et al., 1999) but the size of the scanning window was not discussed. Due to the noise affecting the DEM and to the regional curvature, the local computed curvature may be ambiguous.

The 2D models that account for the width of the flow tube vary in the complexity of their physics and their underlying assumptions (e.g. Martín et al., 2009; Koutnik and Waddington, 2012; Sergienko, 2012). The 2.5D model proposed by Reeh (1988) assumes that the streamlines are perpendicular to the surface contour lines (Fig. 1), so that the mass conservation equation directly depends on the radius of curvature of the surface contour lines (here called R). The model imposes a vertical profile for the horizontal velocity, by the use of a shape function, and the momentum conservation equation was thus not given for any case. This vertical profile is assumed to be unchanging through the x-direction (column-flow model). Later on, Hvidberg (1996) improved upon this approach by setting the stress equilibrium equations without any assumption on the shape of the velocity field. Additionally, when considering a width-varying flow tube, the heat equation should be modified as well, but only few authors have considered this as necessary for their purpose (Hvidberg, 1993, 1996; Pattyn, 2002).

Different authors have used the modelling approach proposed by Reeh (1988), or similar ones, at different scales and for various geometries : for example, mountain glacier (Salamatin et al., 2000; Pattyn, 2002), ice-sheet domes (Reeh and Paterson, 1988; Hvidberg et al., 1997a), ice-sheet ridges (Hvidberg et al., 2002) or even for the whole Greenland (Reeh et al., 2002). The shape and size of the ice bodies are quite diverse, but unfortunately the validity of the 2.5D approach, and its assumptions, for these different cases has received little attention. In particular, the transverse deformation of the ice is assumed to be constant through depth (i.e. the walls of the tube are vertical), which may depend on the surface geometry. Furthermore the error in the computation of R is only discussed by Hvidberg et al. (1997b), who estimated the error in the calculation of R to be 15 %, and Hvidberg et al. (2001) by about 50 %. The method used to measure R is not detailed either, and we doubt that its influence is negligible.

No assumptions of the 2.5D model specifically prohibit its use for a highly diverging tube. However, (Reeh, 1989) determined that the model was incapable of handling such a flow regime, but this was established for a model assuming a constant horizontal velocity profile. Furthermore, Reeh (1989) accounted for the spatial evolution of R with a simple linear model, based on the surface observation. These considerations suggest that domain of applicability of 2.5D models with regards to flow divergence and surface geometry remains to be determined.

As a consequence, the applicability of the 2.5D model should particularly be examined on dome geometries, where simple 2D models would be unable to account for flow divergence. The flow tubes may widen by several orders of magnitude on a few tens of kilometers, especially on the sharpest ridge of the dome. The goal of this study is to perform several comparisons between 2.5D and 3D models, for various dome geometries and temperature conditions, to answer the following questions:

1. What is the error associated with the computation of R from the DEM? This geometric error depends largely on the size of the scanning window.

2. How well can a 3D state of stress be accounted for by the single parameter $R(x)$ along the flow line? This is related to the inherent error of the 2.5 model and its underlying assumptions.

To investigate the performance of the 2.5D model, we compare the velocity fields resulting from the 2.5D and 3D models, the latter being taken as a reference. To our knowledge, no such a systematic comparison between the results of 3D and 2.5D models has been carried out. As such, we hope the present study will guide future research using 2.5D models in different scenarios. In the following, we present the equations of the 3D- and the 2.5D model. We then run the simulations on several domes of different shapes: a circular one (axisymmetric), a slightly elongated one, and a very elongated one. We first consider the isothermal case, before moving on to investigate the effects of temperature.

2 Description of the 3D model

In future work, we hope to investigate a small dome on the East Antarctic plateau. As such, we presently consider a synthetic case with similar geometric and thermal conditions.

2.1 Geometry and mesh

In order to investigate the influence of flow divergence, we model a ridge of a dome, as this results in significant divergence. We perform the present model comparison on a synthetic geometry which consists of a 15 km-radius domain, whose shape is a quarter of cylinder only, for reasons of symmetry. The initial thickness of the ice is 3239 m at the summit, the mean surface slope is around 0.6/1000 and the underlying bed is flat. The space coordinate is a (x, y, z) cartesian system.

The 3D mesh is horizontally unstructured and vertically extruded on 10 levels. The horizontal mean spacing between the nodes is 1 km (Fig. 2).

2.2 Mechanical model

2.2.1 Conservation equations

We denote the velocity vector $\boldsymbol{u}$, with components $(u, v, w)^t$. The stress and strain rate tensors are denoted $\boldsymbol{\sigma}$ and $\dot{\boldsymbol{\epsilon}}$ respectively, and their components, σ_{ij} and $\dot{\epsilon}_{ij}$. The deviatoric part of $\boldsymbol{\sigma}$ is denoted $\boldsymbol{\tau}$, and its components, τ_{ij}. The 3D mechanical model consists of a Stokes problem for incompressible ice of density ρ, in which the mass and momentum conservations equations are written

$$\nabla \cdot \boldsymbol{u} = 0 \tag{1}$$

$$\nabla \cdot \boldsymbol{\sigma} + \rho \boldsymbol{g} = 0 \tag{2}$$

where $\boldsymbol{g}$ is the gravitational acceleration vector. The values of the different parameters are given in Table 1. The ice is assumed to deform following Glen's generalized flow law (Glen, 1958):

$$\dot{\epsilon}_{ij} = A(T)\tau_e^{n-1}\tau_{ij} \tag{3}$$

where τ_e is the second invariant of $\boldsymbol{\tau}$. We choose a value of $n = 3$. The rate factor $A(T)$ non linearly depends on temperature, following an Arrhenius law (Cuffey and Paterson, 2010), reflecting the fact that warm ice is softer. In Antarctica, the temperature range between the surface and the bottom can be large, and the pressure melting point is frequently reached at the bedrock. As a consequence, the viscosity of the upper ice can be two orders of magnitude larger than near the bedrock. To study the influence of the temperature T (expressed in kelvin) on the performance of our 2.5D model, we first consider isothermal ice at $245\,\mathrm{K}$, and then a non-isothermal ice, for which $T(z) = 270 - 50 \cdot (z - b)/(s - b)$, where s and b are the elevation of the surface and the bedrock, respectively. This linear temperature profile is simple but realistic enough to show the effect of a warmer ice at the bottom. For convenience, and as the bed is flat, $b = 0$ in the following experiments.

2.2.2 Boundary conditions

Since the 3D mesh is a quarter of a cylinder, the conditions have to be set on 5 different boundaries, numbered from BC1 to BC5 (Fig. 2). We consider a frozen ice at the bed and no flow through the lateral boundaries (symmetry condition). Considering no sliding at the bottom and neglecting the atmospheric pressure at the surface, the boundary conditions are written as follows:

$$BC1 : \boldsymbol{u}.\boldsymbol{n}|_{y=0} = 0 \tag{4}$$

$$BC2 : \boldsymbol{u}.\boldsymbol{n}|_{x=0} = 0 \tag{5}$$

$$BC3 : \boldsymbol{u}|_{z=0} = \boldsymbol{0} \tag{6}$$

$$BC4 : \boldsymbol{\sigma}.\boldsymbol{n}|_{z=s} = 0 \tag{7}$$

where $\boldsymbol{n}$ is the outward facing normal vector on the surface. Since the surface (BC4) is let free, a kinematic boundary condition for the surface $s(x,y,t)$ has to be solved as well:

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} = w + a \tag{8}$$

where a is the accumulation rate. The velocity boundary condition on BC5 is set to control the shape of the steady-state dome. The method to create an elongated dome is similar to that of Gillet-Chaulet and Hindmarsh (2011), in which the shallow ice approximation is used to prescribe a profile of horizontal velocity at the boundary. As such, BC5 is not a physical boundary, but it allows the domain to be reduced to a reasonable extent. The velocity vector is oriented along the surface slope, and pointing outwards. It is variable with depth to the power $n+1$ and vanishes at the bed. Its norm is tuned depending on its orientation, so that the shape of the dome can be elongated in one preferred direction (Fig. 3):

$$u = \overline{\omega}\,\frac{2\cos\theta}{\alpha}\,\frac{(n+2)}{(n+1)} \times \left(1 - \left(1 - \frac{z}{s}\right)^{(n+1)}\right) \tag{9}$$

$$v = \overline{\omega}\,\frac{2\,(\alpha-1)\sin\theta}{\alpha}\,\frac{(n+2)}{(n+1)} \times \left(1 - \left(1 - \frac{z}{s}\right)^{(n+1)}\right) \tag{10}$$

where θ is the angle to the edge of the domain, and α a shape parameter controlling the elongation of the dome. The following results will correspond to a stabilized steady-state geometry, for a uniform and constant accumulation a in time and space. To do so, the output mean velocity $\overline{\omega}$ is tuned to balance the surface accumulation, so that it can be expressed as

$$\overline{\omega} = \frac{a \cdot \Sigma}{W_L \cdot s} \tag{11}$$

where Σ is the surface area of the accumulation zone on the top boundary (gray area in Fig. 1), and W_L the width of the ice flow at the downstream position $x = L$. In this particular case, Σ and W_L are simply equal to 1/4 of the dome surface and dome perimeter, and L is the radius of the dome. Three different values were taken for the shape parameter α: 2 (axisymmetric case, circular geometry), 3, and 6 (Fig. 3). The axisymmetric domain will test the peformance of the model for a perfectly known case. The latter two cases will test the ability of the model to account for divergence of varying magnitudes.

3 Description of the 2.5 D model

The coordinate system used by Reeh (1988) is a curvilinear coordinate system with right-handed oriented coordinate axis. The x-axis is oriented along the flow line, the z-axis is vertically oriented, and the y-axis is transverse to flow, and tangential to a surface contour line. As we only consider here straight flow lines (linear ridge of an ice divide), the coordinate system is locally cartesian (Fig. 1). We refer to the 2.5D model in the (x,z) coordinate system. We now recall the assumptions made by Hvidberg (1996), partly inherited from Reeh (1988).

1. The flowlines are perpendicular to the surface contour lines.
2. The direction of the horizontal velocity components are constant with depth, which implies that the walls of the flow tube are vertical.
3. There is no shear stresses on the vertical boundaries defined by the flow tube.
4. The ice deforms according to Glen's flow law.

These assumptions together mean that the surface horizontal strain is transferred to the bottom, so that the surface contour lines and the horizontal velocity in the flow direction impose the transverse stresses. Such assumptions are reasonable in the center of an ice-sheet for a slowly varying bed (Reeh, 1988). If the bedrock spatial variations are too steep, they will warp the ice free surface so that the velocity at the base may not be parallel to the velocity at the surface (Hvidberg, 1993; Sergienko, 2012).

3.1 Geometry

The 2D domain is taken as a vertical slice of the 3D domain, on one of its lateral boundaries (Fig. 2). The dome being elongated, we will run the 2.5D model along the sharpest ridge of the 3D dome ($y=0$) or perpendicular to it ($x=0$).

3.2 Mechanical model

We denote the width of the considered flow tube $W(x)$. The radius of the surface contours lines $R(x)$ is taken positive for diverging flow and negative for converging flow. In this model, the assumption (2) implies that W has no dependence on z, so that geometrical considerations show that $W(x)$ is directly linked to $R(x)$ by

$$\frac{1}{R(x)} = \frac{1}{W(x)}\frac{\partial W}{\partial x} \tag{12}$$

An axisymmetric dome leads to the simple relations $R(x) = x$ and $W(x) \propto x$. If the flow tube is diverging more than the axisymmetric flow (on a ridge for example), the corresponding tube surface is narrowed *for a given output width*, and leads to lower output velocities.

The following sections present the equations of mass and momentum conservation, modified to account for the divergence of the tube. As the velocity field mainly depends on the input/output balance, some authors only conserve the mass (e.g., Parrenin et al., 2004; Todd and Christoffersen, 2014), but not the momentum; this approach can in particular be sufficient for a vertically-integrated model (Hvidberg et al., 1997b). Other authors do not modify the Stokes equations at all, but instead add an extra-surface mass balance term (Cook et al., 2014; Gladstone et al., 2012) which depends on the divergence of the tube. This approach has the advantage of simplicity and results in a correct output flux, but neglects the true horizontal advection of the ice. However, this can be justified for ice sheet margins, where the ice mainly undergoes sliding. For all these mass-only conservation models, the normal lateral stress of the surrounding ice is not accounted for, since the force equilibrium is not properly modified.

3.2.1 Mass conservation

At the ice divide, the velocity component v and its spatial derivatives vanish for reasons of symmetry, so that there is no dependence of the strain rates on the transverse coordinate. Under the above assumptions and considering a flow tube of width $W(x)$ (and corresponding radius $R(x)$), the normal strain rates in the curvilinear system are then written (Jaeger, 1969, p. 45):

$$\dot{\epsilon}_{xx} = \frac{\partial u}{\partial x}\,;\, \dot{\epsilon}_{yy} = \frac{u}{R(x)}\,;\, \dot{\epsilon}_{zz} = \frac{\partial w}{\partial z} \tag{13}$$

If the flow tube has a constant width, the value of R is infinite and the equation correspond to the plane strain case. For the more complete form of these expressions, see the discussion of Reeh (1988). The mass conservation then follows:

$$\frac{\partial u}{\partial x} + \frac{u}{R(x)} + \frac{\partial w}{\partial z} = 0 \tag{14}$$

3.2.2 Momentum conservation

For a straight flowline, the horizontal shear strain rate is written (Reeh, 1988):

$$\dot{\epsilon}_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{v}{R(x)} + \frac{\partial v}{\partial x}\right) \tag{15}$$

Along a divide between drainage basins, v vanishes and u attains a local maximum so that $\partial u/\partial y = 0$. In this particular case, there is no horizontal shear strain, and the only non-zero components of strain are the normal ones and the vertical strain rate $\dot{\epsilon}_{xz}$. For dome geometries, Hvidberg (1993) shows that the curvilinear coordinate system is equivalent to a cylindrical coordinate system distorted in the y-direction, for which the radius of curvature $R(x)$ describes the local distortion. The force equilibrium equations, expressed in the (x,z) cartesian coordinate system, are inherited from their formulation in cylindrical coordinates, and are written (Hvidberg, 1996; Jaeger, 1969, p. 123):

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} + \frac{\sigma_{xx} - \sigma_{yy}}{R(x)} = 0 \tag{16}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{xz}}{R(x)} = \rho g \tag{17}$$

where σ_{yy} is known in terms of u and $R(x)$ (Eqs. (3) and (13)).

3.2.3 Boundary conditions

The boundary conditions are inherited from the 3D case: no sliding, free surface, vanishing velocity at the ice divide and an imposed horizontal velocity profile downstream. Note that the value of the mean output velocity $\overline{\omega}$ directly depends on Σ, which is now given by

$$\Sigma = \int_0^L W(x)dx \tag{18}$$

where L is now the length of the flow line. Equations (11) and (18) together mean that the errors in the calculation of W result in errors in the prescribed output velocity. Since we consider a straight flow line, there is no transverse flow across the considered plane. The free surface equation is thus derived from Eq. (8), and is equivalent to a simple 2D case, here given for the $(y = 0)$ plane:

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = w + a \tag{19}$$

3.3 Implementation in Elmer/Ice

The modified mechanical equations are implemented in the Elmer/Ice finite element software (Gagliardini et al., 2013). The correct implementation of the mass conservation was checked by comparing different 2.5D simulations with the Vialov-type profiles (Vialov, 1958) computed for different diverging tubes. The expression of the Vialov profile in the case of a power-law varying flow tube is presented in Appendix A.

3.4 Determination of the contour radius

To determine the radius of curvature of the surface contour lines, we first export a DEM from the surface nodes of the 3D model. These nodes are fitted using an inverse distance weighting, with a power of 4 in order to ensure a good smoothing of the computed surface, representative of a real ice sheet. For comparability with a real case, the spatial resolution of the DEM is taken equal to $400\,\mathrm{m}$, which is the resolution of the DEM resulting from the ICESat mission on Antarctica (Schutz et al., 2005). The DEM surface curvature is computed for each pixel, using a scanning window for fitting (Fig. 1). The local surface inside the window is curve-fitted by a bivariate quadratic polynomial function, and the analytical curvature of this function taken as the local curvature, which depends on the size of the scanning window used. For example, along an elongated ridge

(Fig. 3, middle and right), the surface contours are close to ellipses, but only a sufficiently large scanning window will be able to account for this global shape. A small scanning window may compute an approximately circular curvature. On the contrary, even though a large window may lead to a more accurate value of the curvature geometrically speaking, it is not necessarily the case that it will also lead to a more accurate velocity field, since it is difficult to know which surrounding environment is mainly influencing the ice flow. As a consequence, three different sizes of scanning window are here tested: $2.8\,\mathrm{km}$, $6\,\mathrm{km}$ and $10\,\mathrm{km}$, thus corresponding to a width of 7, 15 and 25 pixels. For each pixel, the value of $R(x)$ is taken as the inverse of the curvature of the contours of the fitted surface in the center of the window. This is done using GRASS GIS software.

3.5 Protocol of comparison

We first run the 3D transient isothermal simulation, and stop when a steady state is reached ($\partial s/\partial t < 10^{-6}\,\mathrm{m/a}$). We then use the resulting DEM to compute the profile of R to initialize the 2.5D model. For each dome configuration (α = 2,3,6) we compare the different runs chosen: a) 3D, b) 2.5D for the three scanning windows, and a fixed geometry, c) 2.5D for the three scanning windows, with a free surface. For the axi-symmetric case we add a true 2.5D axisymmetric run (imposing an analytical value $R = x$). Then we compare the 3D and 2.5D results for a non-isothermal ice, to investigate the influence of the temperature, especially near the base of the ice sheet.

4 Results and discussion

4.1 Circular geometry ($\alpha = 2$), isothermal ice

4.1.1 3D/2.5D axisymmetric comparison

The absolute error on the ice velocity for an axisymmetric 2.5D model ($R = x$) is of the order of $10^{-4}\,\mathrm{m/a}$ (Fig. 4.a, where the black and yellow curves are almost superimposed). The observed error is a result of discretization, and should tend to zero as the element sizes decrease.

4.1.2 3D/2.5D comparison

The computation of the radius of the surface contour lines is strongly influenced by the size of the scanning window. For a circular geometry, the variation of W along x should be linear, which is almost the case for the two wider windows (Fig. 5, solid lines). With this regular geometry, the larger the window, the more precise the radius, since the fitted surface will be more accurate. On the contrary, the width value computed with the smaller window is less regular and underestimated by about 30%, meaning that we cannot evaluate a certain curvature from too small a sample. When choosing the intermediate or the larger scanning-window, the root mean square error (RMSE) in the velocity is 9.9 % or 3.1 % respectively (Fig. 4.a), and is a consequence of the error in the calculation of the radius.

4.2 Elongated domes ($\alpha = 3$ and $\alpha = 6$), isothermal ice

For elongated domes, we consider both the flow line along the sharpest ridge ($y = 0$), and perpendicular to it ($x = 0$).

4.2.1 3D/2.5D comparison, along the ridge

Along a ridge, the flow tube is non linearly diverging. For a given output width, the accumulation area is smaller than in the axisymmetric case, thus leading to lower output velocities.

With fixed geometries, it clearly appears that the velocity is underestimated for elongated domes (Fig. 4.b and 4.c, dashed lines), meaning that the local surface slope cannot explain the ice motion by itself: the ice along the ridge is also, if not mainly, pulled by the surrounding lateral ice, which moves due to a steeper surface slope. The case of the small scanning window appears to be different (Fig. 4.b, red dashed line), simply because of a really poor estimation of W, thus of the output velocity.

The downstream velocities are always quite accurate (10 % error), since they mainly depend on the tube surface calculation, incorporated in the velocity boundary condition. When releasing the surface, the surface slope slightly increases to accomodate the velocity boundary condition, and the computed velocity field is then closer to the 3D reference. The relative error made in the downstream part of the flow is comparatively higher near the divide since the velocities are very small.

In case of a sharp ridge, the ice surface has the shape of a circus tent (Fig. 6, dotted line), i.e. the ice slope increases when going towards the divide. This shape is clearly not physical, and is the result of a numerical artefact. Since the vertical strain rate is always of the order of a/H, the conservation equation leads to a balance between $\partial u/\partial x$ and u/R close to the divide. For highly diverging flows, u/R is much smaller than for an axisymmetric case at the same x position (Appendix B). As a consequence, $\partial u/\partial x$ should have at a comparatively much higher value. The only way for u to increase over a short distance near the divide is for the surface elevation to decrease sharply. To handle this artefact, we increased the mesh resolution near the dome, but without any successful results. This artefact does not appear for $\alpha = 3$ (slightly elongated dome).

For $\alpha = 6$, the tube surface is better estimated with an intermediate window, whose size is closer to the local value of R. The RMSE in the velocity is 12.1 % for the intermediate window and 44.3 % for the large window. Too large a window would consider the whole shape of the dome and lead to an underestimation of R. The amplitude of the error between the different runs show that for sharp ridges (or highly diverging tubes) the choice of the window size is not straightforward, as a wider window increases the regularity of the velocity field, but decreases the ability to capture the local curvature.

Reeh (1989), using a 2.5D model for dating purposes, explained the large errors for the diverging tube of Camp Century partly by the simplicity of his model, especially his linear model of R. The computed velocities show important discrepancy with the oberved ones, which leads to a bad estimation of the origin position of the ice, by 200 %. The error made in the velocity field with this more complete model seems now to be small enough for tracking ice particles correctly, for example with the intermediate window size. However, for a real case of highly curved surface, it is difficult to know *a priori* the best window size to use and if the divergence can be properly accounted for, in the absence of a reference as it is done here.

4.2.2 3D/2.5D comparison, perpendicular to the ridge

As the divergence is much smaller perpendicular to the sharpest ridge, the velocity field is much smoother, and its spatial evolution closer to the 3D reference than the along-ridge case. The RMSE is 11.9 % and 7.5 % for the intermediate and large window size respectively, for $\alpha = 3$, and 13.7 % and 11.1 % for $\alpha = 6$; this error is slightly higher than for the circular geometry (Fig. 4.d). In the case of large radius values, towards the exterior of the dome, the wider window gives the more accurate results.

4.3 Non-isothermal ice

A supplementary comparison is carried out on the sharp ridge ($y = 0$) for temperature varying linearly through depth. The computed velocity field towards the divide (low x values) shows a reversed vertical profile, i.e. the basal ice goes faster than the upper ice (Fig. 7, bottom). This non-physical result in 2.5D can be explained this way: as soon as a 3D tube diverges more than for an axisymmetric flow, the warmer basal ice is more easily laterally strained than the colder surface ice. As a consequence, the walls of the 3D flow tube are no longer vertical, and using the 2.5D model in such a case would violate assumption (2). This effect is particularly pronounced close to the divide, where the tube is narrower, and can be seen in the 3D simulations as follows (Fig. 8). The stream lines going through a flux gate at $x = 1000\,\mathrm{m}$ are tracked on a few hundred metres ($x = 1200\,\mathrm{m}$ for $\alpha = 6$, $x = 1500\,\mathrm{m}$ for $\alpha = 3$). The divergence of the stream lines is depending on their depth – the tube is larger near the bottom, Fig. 8, blue curves –, and this dependence is stronger for high diverging tubes. To accomodate the lateral strain in this case, the 2.5D model computes high horizontal velocities in the bottom, whereas the real motion is in fact mainly laterally oriented. No mesh refinement has been able to correct this problem. Since it does not happen for a constant temperature, it certainly originates from the lower viscosity of the basal ice (Fig. 8, red curves).

This result also suggests that, on sharp ridges and with non-isothermal ice, working with a fixed vertical profile of velocity will prevent from such unintended behaviour. This artefact may affect the results of Hvidberg et al. (2002), who study a flow line between GRIP (Summit) and North GRIP, because their model accounts for temperature. Their flowline stretches along a ridge which can be quite sharp, and for which the flow divergence is probably higher than axisymmetry. However, it is much longer than ours, and the ridge is the sharpest far from the summit, whereas it is the contrary in our case. The artefact that appears in our simulations may not be as sensitive in their case. Nevertheless, care must be taken in such cases, since the basic assumptions may not be justifiable and the model is likely to be outwith its application domain.

For reasons of continuity, the walls of the flow tube cannot be vertical in the direction perpendicular to the ridge either, but the effect is too weak to impact the computed velocity field.

5 Conclusions

A systematic comparison between 2.5D- and 3D models has been presented in order to evaluate the ability of the former to accurately compute the velocity field on a small dome of an ice sheet. The error made when estimating the value of the radius of the surface contour lines is of the order of 10% if the computation window is well chosen, though it can be comparatively higher close to the divide. The radius of curvature of the surface elevation contour lines should be determined with a sufficiently large computation window, but choosing the optimum size is not completely straightforward; in any case we suggest it should not be less than one-third of the maximum measured radius, and several windows should be tested to ensure the robustness of the results.

The 2.5D model can be used without any specific restriction for tubes diverging less than and up to an axisymmetric flow. For isothermal ice, the model can be used with tubes diverging more than an axisymmetric flow, if the divergence is not too high. For very high divergence, the ice is in our study mainly pulled by the output boundary condition, and the resulted velocity field may be somewhat irregular, and surface geometry unphysical close to the divide. This means that, in the case of a sharp ridge, accounting for a certain state of stress via a single parameter $R(x)$ is clearly not possible. In practice, the examples of surface topographies given in this study may serve as a reference to evaluate if using a 2.5D model for a real case is justifiable.

For non isothermal ice, the tube should not diverge more than axisymmetry, because the softer basal ice would be much more easily laterally strained in the case of an elongated dome. The walls of the flow tube are therefore not vertical, which violates the model assumptions, and the corresponding horizontal velocity profile may be not physical near the divide. This has significant consequences for dating purposes: as the computed velocity field shows too small values in the upper layers, the age of the ice is overestimated, and the modelled isochrone layers are too high. In the absence of reference 3D solution as produced here, the velocity field may not necessarily appear to be unphysical, but this does not mean that the numerical artefact does not significantly affect the age calculation.

This study shows that the use of a 2.5D model, which is a trade off between 2D and 3D, must avoid several pitfalls, and its applicability domain appears to be significantly narrower than initially thought. The geometric error, resulting from miscalculation of R, should have less impact on the quality of the results than the inherent errors related to the violation of the 2.5D model's assumptions, especially when the temperature varies through the ice column. The 2.5D model was designed for divides between drainage basins, but real flowlines along sharp divides should not be modelled this way. Finally, for a real case, it has not been established how flat the bedrock should be to ensure that the assumption of verticality of the walls of the flow tube is reasonably respected, even for a slightly divergent flow. The applicability domain of the 2.5D model for slightly horizontally curved flowlines has not been investigated here, but further uncertainties are expected in this case, for which some secondary-order terms are neglected in the mass conservation.

Code availability

The presented simulations were performed using the finite element model Elmer/Ice v.7.0 rev. 7016. The source code of the 2.5D model is available in the distribution since v.8.0 rev. d9d4a2f, implemented in the AIFlow solver.

Author contribution

The experiments were designed by OG, FP and JT. OP carried them out, helped by CR for analytical developments, and by FGC for the Elmer/Ice implementation. OP prepared the manuscript with contributions from all co-authors.

Appendix A: Vialov profile for a power-law diverging tube

To check the correct implementation of the mass conservation in the 2.5D model, we hereafter compute the height of a Vialov profile corresponding to a regularly diverging flow tube. Note that such a surface is only representative of a single flow line, and not for a whole surface, as what is usually done for a Vialov profile in plane strain (Vialov, 1958) or axisymmetry (Ritz, 1992).

Figure 5 shows that we may approximate the shape of the flow tube by a power-law depending on the x-coordinate. Let consider a flow tube of width $W = W_L \left(\frac{x}{L}\right)^{\beta}$. For plane flow, $\beta = 0$, for axisymmetry $\beta = 1$, and for sharp ridges $\beta > 1$. The volume outflow q^* for a certain coordinate x may be expressed in one of two ways (Cuffey and Paterson, 2010, p.388):

$$q^* = \int_0^x a\left(\frac{x'}{L}\right)^{\beta} W_L \, dx' = \frac{ax}{(\beta+1)}\left(\frac{x}{L}\right)^{\beta} W_L = \frac{axW}{(\beta+1)} \tag{A1}$$

$$q^* = \left(\frac{2A}{n+2}\tau_b^n H\right) HW \tag{A2}$$

where H is the ice thickness, τ_b the basal shear, a the accumulation rate, L the length of the glacier, A the rate factor of Glen's flow law. Equating the two expressions yields

$$a \cdot x = \frac{2A(\beta+1)}{n+2}\tau_b^n H^2 \tag{A3}$$

The following reasoning is then similar to that of Cuffey and Paterson (2010), simply modified by a $(\beta+1)$ multiplier. The final expression for the ice thickness is unchanged

$$H = H^* \left(1 - \left(\frac{x}{L}\right)^{\frac{n+1}{n}}\right)^{\frac{n}{2n+2}} \tag{A4}$$

except the height of the ice sheet H^* at the dome ($x = 0$), which is now

$$H^* = \left(\frac{2(n+2)^{1/n}}{\rho g}\right)^{\frac{n}{2n+2}} \sqrt{L}\left(\frac{a}{2A(\beta+1)}\right)^{\frac{1}{2n+2}} \tag{A5}$$

This expression is consistent with the one previously derived for axisymmetry. We then use this expression to control the 2.5D model by comparing the value of H^* computed by the model with its above theoretical value.

Appendix B: Radius and surface of a power-law diverging flow tube

We consider the same flow as in Appendix A. The value of the radius $R(x)$ is then expressed as

$$\frac{1}{R(x)} = \frac{1}{W}\frac{dW}{dx} = \frac{dln(W)}{dx} = \frac{\beta}{x} \tag{B1}$$

The surface area of the tube upstream of x can be expressed as

$$\Sigma(x) = \int_0^x W(x)dx = \frac{W_L}{\beta+1}\frac{x^{\beta+1}}{L^\beta} \tag{B2}$$

As u is more or less proportionnal to the upstream surface area Σ, u/R is expected to be proportionnal to $W_L\left(\frac{x}{L}\right)^\beta$. On the contrary, one can consider that the value of $\partial u/\partial x$ should be of the order of $\overline{\omega}/L$, i.e. simply proportionnal to $\Sigma(L)/L = W_L/(\beta+1)$. Near the divide, u/R is then comparatively much smaller than $\partial u/\partial x$ for sharp ridges than for axisymmetric flows, and imbalances the corresponding mass conservation.

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Table 1. Description and values of the model parameters.

Parameter	Variable	Value	Unit
Accumulation rate	a	0.04	$\mathrm{m/a}$
Flow line length	L	15 000	m
Downstream width	W_L	1	m
Flow law exponent	n	3.0	
Initial max. ice thickness	max(s)	3239	m
Ice density	ρ	917	$\mathrm{kg/m^3}$
Mean surface slope		0.6 / 1000	

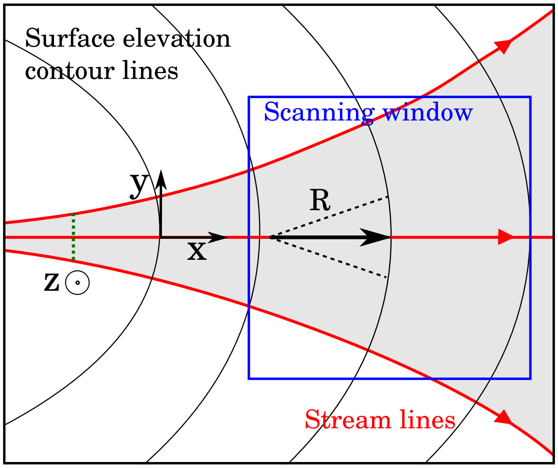

Figure 1. The x-axis is taken along the ridge stream line, and the y-axis tangent to the surface elevation contour. The scanning window is used to determine the value of R in its center. Lateral stream lines are represented to show the non linear widening of the flow tube in this particular case, and the corresponding accumulation surface is colored in gray.

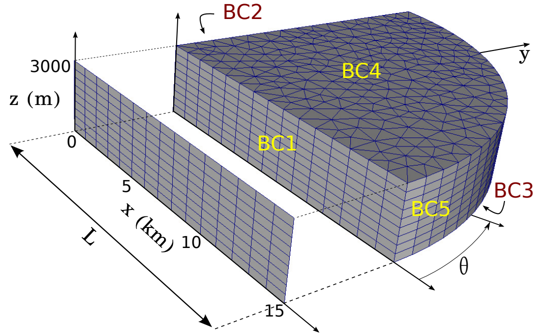

Figure 2. View of the two meshes used in this study. For each run, the 2D mesh is extracted from the geometry of the steady state solution of the 3D simulation. BC1 to BC5 refer to the 5 boundary conditions of the 3D case.

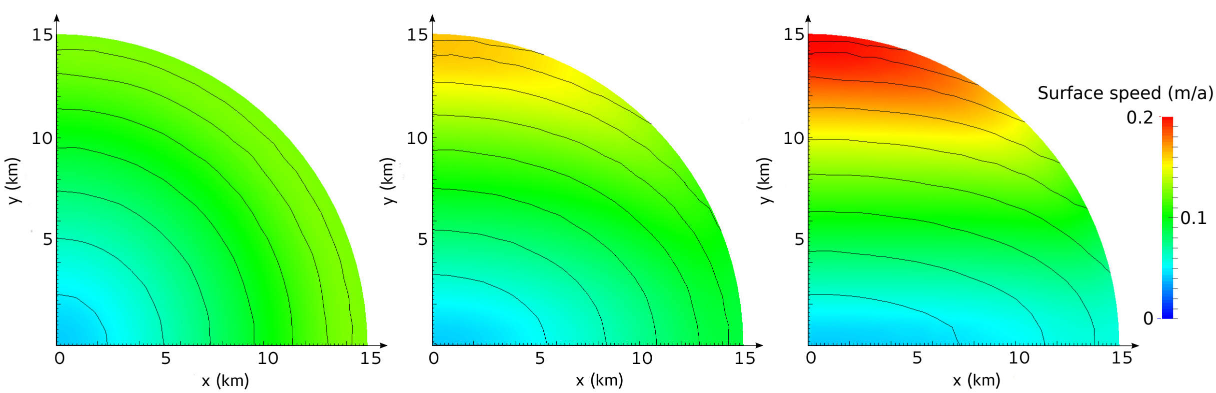

Figure 3. The three domes stabilized at steady state: surface contour lines (spacing: $1\,\mathrm{m}$), and surface speed $(\mathrm{m/a})$ for $\alpha = 2$ (left), $\alpha = 3$ (middle) and $\alpha = 6$ (right).

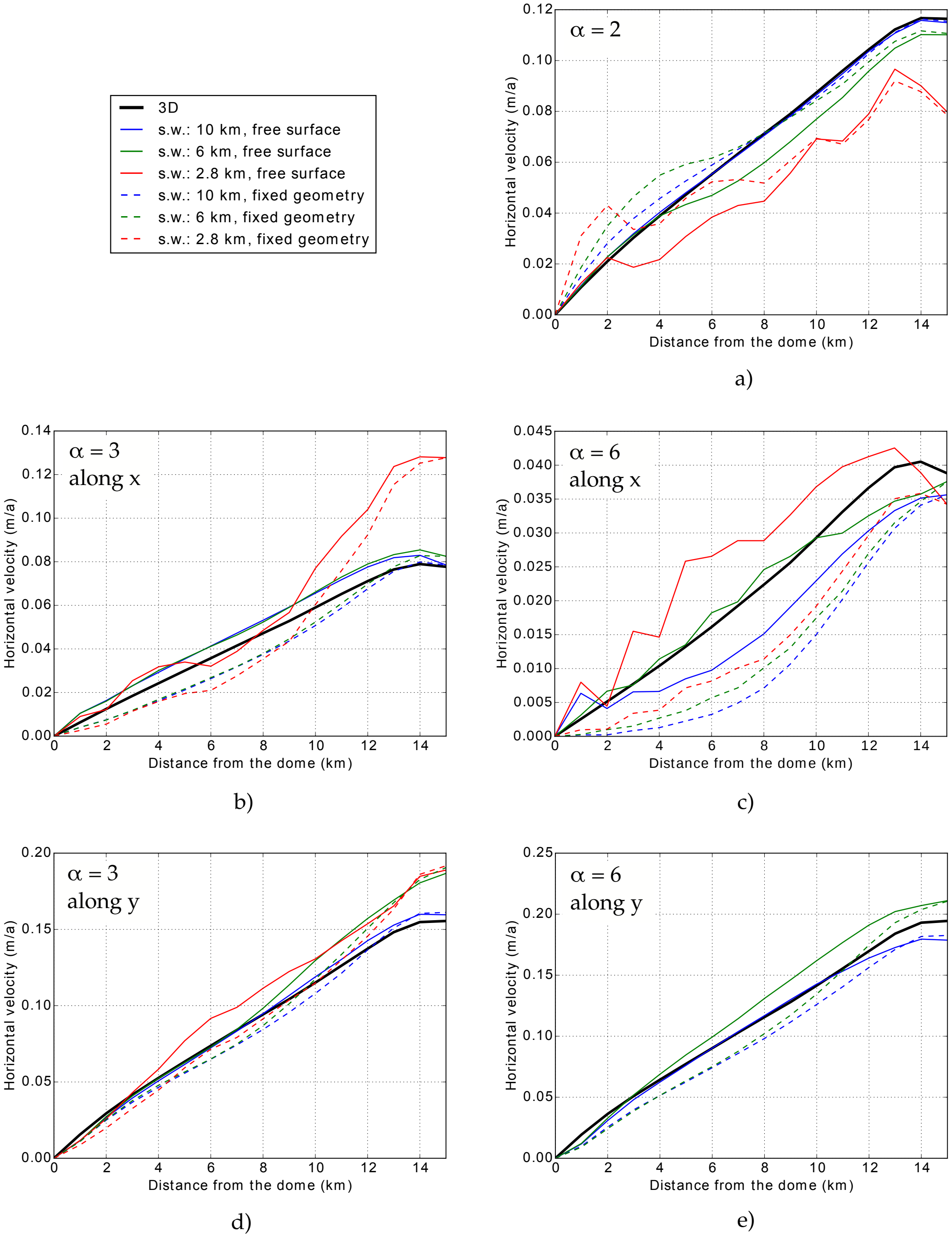

Figure 4. Horizontal velocity at the ice surface $(\mathrm{m/a})$ along the flow line, for isothermal ice. Dashed lines are for fixed geometry runs, and solid lines for evolved free surface to a steady state. Thick black : 3D; yellow: 2.5D axisymmetric (hidden by the black curve); red: 2.5D scanning window of 2.8 km; green: 2.5D scanning window of 6 km; blue: 2.5D scanning window of 10 km. a) Circular dome, $\alpha = 2$. b) $\alpha = 3$, along the ridge. c) $\alpha = 6$, along the ridge. d) $\alpha = 3$, perpendicular to the ridge. e) $\alpha = 6$, perpendicular to the ridge. For this last case, no result concerning the 2.8 km scanning window is shown since they are completely out of reasonable bounds.

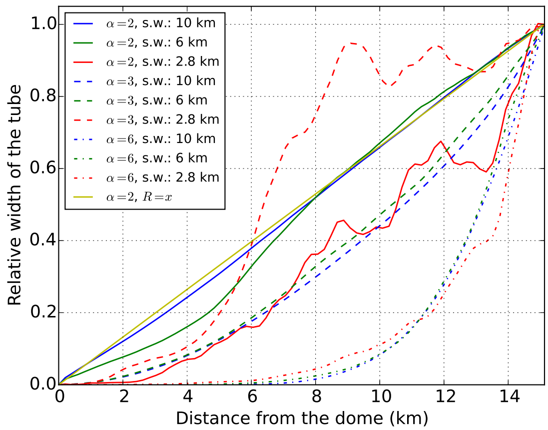

Figure 5. Relative width of the flow tube W for the different dome geometries: circular ($\alpha = 2$, solid lines), slightly elongated ($\alpha = 3$, dashed lines) and very elongated ($\alpha = 6$, dotted lines); for different scanning windows: 2.8 km (red), 6 km (green) and 10 km (blue). The yellow curve is the line for which $R = x$ is imposed (axisymmetric case).

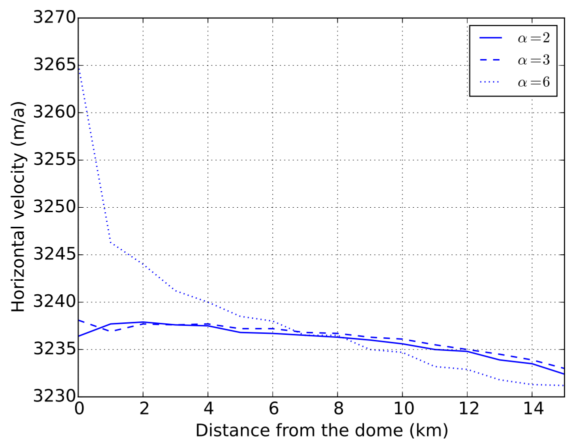

Figure 6. Height of the ice surface (m) for the 2.5D free surface model, along the ridge. Solid line: circular geometry, $\alpha = 2$. Dashed line: $\alpha = 3$. Dotted line: $\alpha = 6$.

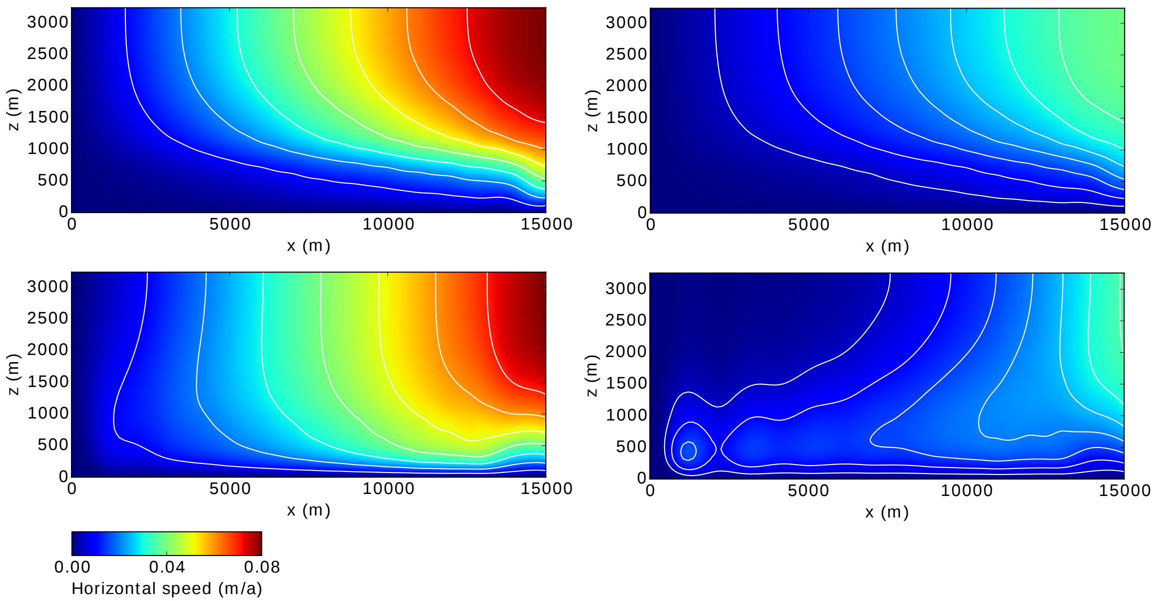

Figure 7. Horizontal velocity field (m/a), for the non isothermal case and $\alpha = 3$. Top: 3D model. Bottom: 2.5D model, with a scanning window of $10\,\mathrm{km}$. Left: $\alpha = 3$, white contour lines spaced by $0.01\,\mathrm{m/a}$. Right: $\alpha = 6$, white contour lines spaced by $0.005\,\mathrm{m/a}$.

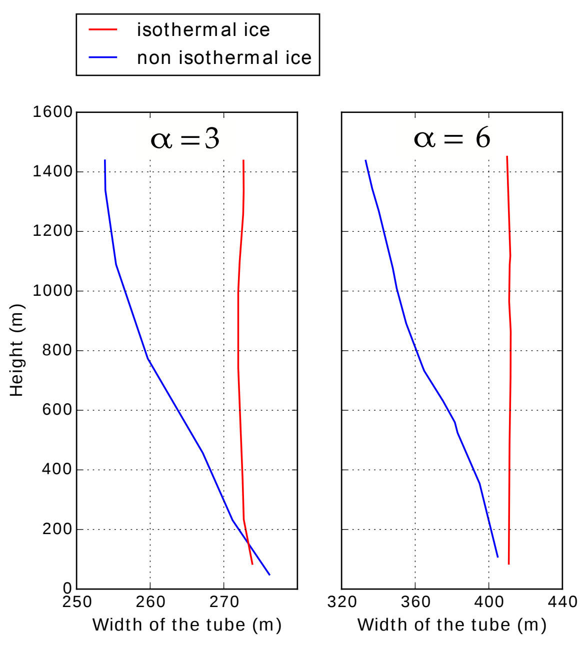

Figure 8. Width of the flow tube (m), for the isothermal (red) and non isothermal case (blue), for $\alpha = 3$ ($x = 1500\,\mathrm{m}$, left) and $\alpha = 6$ ($x = 1200\,\mathrm{m}$, right) The aspect of the curves is slightly affected by numerical noise due to discretization.