S1 Functional Form of a Quadrivariate PDF

There is one type of quadrivariate PDF used in the equation set. It is quadrivariate normal-normal-lognormal-lognormal distribution, meaning that the individual marginal of x_1 is a normal distribution, the individual marginal of x_2 is a normal distribution, the individual marginal of x_3 is a lognormal distribution, and the individual marginal of x_4 is a lognormal distribution. The functional form of this type of PDF is given by:

$$P_{NNLL(i)}\left(x_1, x_2, x_3, x_4\right) = \frac{\exp\left\{-\frac{1}{2}\lambda_{NNLL}\right\}}{(2\pi)^2\,\sigma_{x_1(i)}\sigma_{x_2(i)}\tilde{\sigma}_{x_3(i)}\tilde{\sigma}_{x_4(i)}C_{Q1}x_3x_4}; \qquad \text{(S1)}$$

where:

$$\begin{aligned}\lambda_{NNLL} = \frac{1}{C_{Q1}^2}\Big[& C_{Q2}\left(x_1-\mu_{x_1(i)}\right)^2 + C_{Q3}\left(x_2-\mu_{x_2(i)}\right)^2 + C_{Q4}\left(\ln x_3-\tilde{\mu}_{x_3(i)}\right)^2 \\ & + C_{Q5}\left(\ln x_4-\tilde{\mu}_{x_4(i)}\right)^2 + C_{Q6}\left(x_1-\mu_{x_1(i)}\right)\left(x_2-\mu_{x_2(i)}\right) \\ & + C_{Q7}\left(x_1-\mu_{x_1(i)}\right)\left(\ln x_3-\tilde{\mu}_{x_3(i)}\right) + C_{Q8}\left(x_1-\mu_{x_1(i)}\right)\left(\ln x_4-\tilde{\mu}_{x_4(i)}\right) \\ & + C_{Q9}\left(x_2-\mu_{x_2(i)}\right)\left(\ln x_3-\tilde{\mu}_{x_3(i)}\right) + C_{Q10}\left(x_2-\mu_{x_2(i)}\right)\left(\ln x_4-\tilde{\mu}_{x_4(i)}\right) \\ & + C_{Q11}\left(\ln x_3-\tilde{\mu}_{x_3(i)}\right)\left(\ln x_4-\tilde{\mu}_{x_4(i)}\right)\Big];\end{aligned}$$

and where:

$$\begin{aligned}C_{Q1} = \Big[& 1-\left(\rho_{x_1,x_2(i)}^2+\tilde{\rho}_{x_1,x_3(i)}^2+\tilde{\rho}_{x_1,x_4(i)}^2+\tilde{\rho}_{x_2,x_3(i)}^2+\tilde{\rho}_{x_2,x_4(i)}^2+\tilde{\rho}_{x_3,x_4(i)}^2\right) \\ & + 2\rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_3(i)} + 2\rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_2,x_4(i)} \\ & + 2\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_3,x_4(i)} + 2\tilde{\rho}_{x_2,x_3(i)}\tilde{\rho}_{x_2,x_4(i)}\tilde{\rho}_{x_3,x_4(i)} + \rho_{x_1,x_2(i)}^2\tilde{\rho}_{x_3,x_4(i)}^2 \\ & + \tilde{\rho}_{x_1,x_3(i)}^2\tilde{\rho}_{x_2,x_4(i)}^2 + \tilde{\rho}_{x_1,x_4(i)}^2\tilde{\rho}_{x_2,x_3(i)}^2 - 2\rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_4(i)}\tilde{\rho}_{x_3,x_4(i)} \\ & - 2\rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_2,x_3(i)}\tilde{\rho}_{x_3,x_4(i)} - 2\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_2,x_3(i)}\tilde{\rho}_{x_2,x_4(i)}\Big]^{\frac{1}{2}};\end{aligned}$$

$$C_{Q2} = \frac{1}{\sigma_{x_1(i)}^2}\Big[1-\left(\tilde{\rho}_{x_2,x_3(i)}^2+\tilde{\rho}_{x_2,x_4(i)}^2+\tilde{\rho}_{x_3,x_4(i)}^2\right)+2\tilde{\rho}_{x_2,x_3(i)}\tilde{\rho}_{x_2,x_4(i)}\tilde{\rho}_{x_3,x_4(i)}\Big];$$

$$C_{Q3} = \frac{1}{\sigma^2_{x_2(i)}}\Big[1 - \big(\tilde{\rho}^2_{x_1,x_3(i)} + \tilde{\rho}^2_{x_1,x_4(i)} + \tilde{\rho}^2_{x_3,x_4(i)}\big) + 2\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_3,x_4(i)}\Big];$$

$$C_{Q4} = \frac{1}{\tilde{\sigma}^2_{x_3(i)}}\Big[1 - \big(\rho^2_{x_1,x_2(i)} + \tilde{\rho}^2_{x_1,x_4(i)} + \tilde{\rho}^2_{x_2,x_4(i)}\big) + 2\rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_2,x_4(i)}\Big];$$

$$C_{Q5} = \frac{1}{\tilde{\sigma}^2_{x_4(i)}}\Big[1 - \big(\rho^2_{x_1,x_2(i)} + \tilde{\rho}^2_{x_1,x_3(i)} + \tilde{\rho}^2_{x_2,x_3(i)}\big) + 2\rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_3(i)}\Big];$$

$$\begin{aligned} C_{Q6} = \frac{2}{\sigma_{x_1(i)}\sigma_{x_2(i)}}\Big(&\rho_{x_1,x_2(i)}\tilde{\rho}^2_{x_3,x_4(i)} - \tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_2,x_3(i)}\tilde{\rho}_{x_3,x_4(i)} \\ &- \tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_4(i)}\tilde{\rho}_{x_3,x_4(i)} + \tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_3(i)} + \tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_2,x_4(i)} \\ &- \rho_{x_1,x_2(i)}\Big); \end{aligned}$$

$$\begin{aligned} C_{Q7} = \frac{2}{\sigma_{x_1(i)}\tilde{\sigma}_{x_3(i)}}\Big(&\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}^2_{x_2,x_4(i)} - \rho_{x_1,x_2(i)}\tilde{\rho}_{x_2,x_4(i)}\tilde{\rho}_{x_3,x_4(i)} \\ &- \tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_2,x_3(i)}\tilde{\rho}_{x_2,x_4(i)} + \rho_{x_1,x_2(i)}\tilde{\rho}_{x_2,x_3(i)} + \tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_3,x_4(i)} \\ &- \tilde{\rho}_{x_1,x_3(i)}\Big); \end{aligned}$$

$$\begin{aligned} C_{Q8} = \frac{2}{\sigma_{x_1(i)}\tilde{\sigma}_{x_4(i)}}\Big(&\tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}^2_{x_2,x_3(i)} - \rho_{x_1,x_2(i)}\tilde{\rho}_{x_2,x_3(i)}\tilde{\rho}_{x_3,x_4(i)} \\ &- \tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_3(i)}\tilde{\rho}_{x_2,x_4(i)} + \rho_{x_1,x_2(i)}\tilde{\rho}_{x_2,x_4(i)} + \tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_3,x_4(i)} \\ &- \tilde{\rho}_{x_1,x_4(i)}\Big); \end{aligned}$$

$$\begin{aligned} C_{Q9} = \frac{2}{\sigma_{x_2(i)}\tilde{\sigma}_{x_3(i)}}\Big(&\tilde{\rho}^2_{x_1,x_4(i)}\tilde{\rho}_{x_2,x_3(i)} - \rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_3,x_4(i)} \\ &- \tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_2,x_4(i)} + \tilde{\rho}_{x_2,x_4(i)}\tilde{\rho}_{x_3,x_4(i)} + \rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_3(i)} \\ &- \tilde{\rho}_{x_2,x_3(i)}\Big); \end{aligned}$$

$$\begin{aligned} C_{Q10} = \frac{2}{\sigma_{x_2(i)}\tilde{\sigma}_{x_4(i)}}\Big(&\tilde{\rho}^2_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_4(i)} - \rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_3,x_4(i)} \\ &- \tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_2,x_3(i)} + \tilde{\rho}_{x_2,x_3(i)}\tilde{\rho}_{x_3,x_4(i)} + \rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_4(i)} \\ &- \tilde{\rho}_{x_2,x_4(i)}\Big); \end{aligned}$$

$$C_{Q11} = \frac{2}{\tilde{\sigma}_{x_3(i)}\tilde{\sigma}_{x_4(i)}}\Big(\rho^2_{x_1,x_2(i)}\tilde{\rho}_{x_3,x_4(i)} - \rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_2,x_3(i)}$$

$$- \rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_4(i)} + \tilde{\rho}_{x_2,x_3(i)}\tilde{\rho}_{x_2,x_4(i)} + \tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_1,x_4(i)}$$

$$- \tilde{\rho}_{x_3,x_4(i)}\Big).$$

In Eq. (S1), $\mu_{x_1(i)}$ is the mean of x_1 in the ith component, $\mu_{x_2(i)}$ is the mean of x_2 in the ith component, $\tilde{\mu}_{x_3(i)}$ is the mean of $\ln x_3$ in the ith component, and $\tilde{\mu}_{x_4(i)}$ is the mean of $\ln x_4$ in the ith component. The ith component standard deviation of x_1 is $\sigma_{x_1(i)}$, the ith component standard deviation of x_2 is $\sigma_{x_2(i)}$, the ith component standard deviation of $\ln x_3$ is $\tilde{\sigma}_{x_3(i)}$, and the ith component standard deviation of $\ln x_4$ is $\tilde{\sigma}_{x_4(i)}$. The ith component correlation of x_1 and x_2 is $\rho_{x_1,x_2(i)}$, the ith component correlation of x_1 and $\ln x_3$ is $\tilde{\rho}_{x_1,x_3(i)}$, the ith component correlation of x_1 and $\ln x_4$ is $\tilde{\rho}_{x_1,x_4(i)}$, the ith component correlation of x_2 and $\ln x_3$ is $\tilde{\rho}_{x_2,x_3(i)}$, the ith component correlation of x_2 and $\ln x_4$ is $\tilde{\rho}_{x_2,x_4(i)}$, and the ith component correlation of $\ln x_3$ and $\ln x_4$ is $\tilde{\rho}_{x_3,x_4(i)}$.

S2 Functional Form of Trivariate PDFs

There are two types of trivariate PDFs used in the equation set. The first one is a trivariate normal-normal-lognormal distribution, meaning that the individual marginal of x_1 is a normal distribution, the individual marginal of x_2 is a normal distribution, and the individual marginal of x_3 is a lognormal distribution. The functional form of this type of PDF is given by:

$$P_{NNL(i)}\left(x_1, x_2, x_3\right) = \frac{\exp\left\{-\frac{1}{2}\lambda_{NNL}\right\}}{(2\pi)^{\frac{3}{2}}\,\sigma_{x_1(i)}\sigma_{x_2(i)}\tilde{\sigma}_{x_3(i)}C_{T1}x_3}; \tag{S2}$$

where:

$$\lambda_{NNL} = \frac{1}{C_{T1}^2}\Big[C_{T2}\left(x_1-\mu_{x_1(i)}\right)^2 + C_{T3}\left(x_2-\mu_{x_2(i)}\right)^2 + C_{T4}\left(\ln x_3-\tilde{\mu}_{x_3(i)}\right)^2$$
$$+ C_{T5}\left(x_1-\mu_{x_1(i)}\right)\left(x_2-\mu_{x_2(i)}\right) + C_{T6}\left(x_1-\mu_{x_1(i)}\right)\left(\ln x_3-\tilde{\mu}_{x_3(i)}\right)$$
$$+ C_{T7}\left(x_2-\mu_{x_2(i)}\right)\left(\ln x_3-\tilde{\mu}_{x_3(i)}\right)\Big];$$

and where:

$$C_{T1} = \left[1-\left(\rho_{x_1,x_2(i)}^2+\tilde{\rho}_{x_1,x_3(i)}^2+\tilde{\rho}_{x_2,x_3(i)}^2\right)+2\rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_3(i)}\right]^{\frac{1}{2}};$$

$$C_{T2} = \frac{1-\tilde{\rho}_{x_2,x_3(i)}^2}{\sigma_{x_1(i)}^2}; \qquad C_{T3} = \frac{1-\tilde{\rho}_{x_1,x_3(i)}^2}{\sigma_{x_2(i)}^2}; \qquad C_{T4} = \frac{1-\rho_{x_1,x_2(i)}^2}{\tilde{\sigma}_{x_3(i)}^2};$$

$$C_{T5} = \frac{2\left(\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_3(i)}-\rho_{x_1,x_2(i)}\right)}{\sigma_{x_{1(i)}}\sigma_{x_{2(i)}}}; \quad C_{T6} = \frac{2\left(\rho_{x_1,x_2(i)}\tilde{\rho}_{x_2,x_3(i)}-\tilde{\rho}_{x_1,x_3(i)}\right)}{\sigma_{x_{1(i)}}\tilde{\sigma}_{x_{3(i)}}};$$

and $$C_{T7} = \frac{2\left(\rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_3(i)}-\tilde{\rho}_{x_2,x_3(i)}\right)}{\sigma_{x_{2(i)}}\tilde{\sigma}_{x_{3(i)}}}.$$

In Eq. (S2), $\mu_{x_1(i)}$ is the mean of x_1 in the ith component, $\mu_{x_2(i)}$ is the mean of x_2 in the ith component, and $\tilde{\mu}_{x_3(i)}$ is the mean of $\ln x_3$ in the ith component. The ith component standard deviation of x_1 is $\sigma_{x_1(i)}$, the ith component standard deviation of x_2 is $\sigma_{x_2(i)}$, and the ith component standard deviation of $\ln x_3$ is $\tilde{\sigma}_{x_3(i)}$. The ith component correlation of x_1 and x_2 is $\rho_{x_1,x_2(i)}$, the ith component correlation of x_1 and $\ln x_3$ is $\tilde{\rho}_{x_1,x_3(i)}$, and the ith component correlation of x_2 and $\ln x_3$ is $\tilde{\rho}_{x_2,x_3(i)}$.

The second type of trivariate PDF used in the equation set is a trivariate normal-lognormal-lognormal distribution, meaning that the individual marginal of x_1 is a normal distribution, the individual marginal of x_2 is a lognormal distribution, and the individual marginal of x_3 is a lognormal distribution. The functional form of this type of PDF is given by:

$$P_{NLL(i)}\left(x_1,x_2,x_3\right)=\frac{\exp\left\{-\frac{1}{2}\lambda_{NLL}\right\}}{(2\pi)^{\frac{3}{2}}\,\sigma_{x_1(i)}\tilde{\sigma}_{x_2(i)}\tilde{\sigma}_{x_3(i)}C_{t1}x_2x_3}; \tag{S3}$$

where:

$$\begin{aligned}\lambda_{NLL}=\frac{1}{C_{t1}^2}\Big[&C_{t2}\left(x_1-\mu_{x_1(i)}\right)^2+C_{t3}\left(\ln x_2-\tilde{\mu}_{x_2(i)}\right)^2+C_{t4}\left(\ln x_3-\tilde{\mu}_{x_3(i)}\right)^2\\&+C_{t5}\left(x_1-\mu_{x_1(i)}\right)\left(\ln x_2-\tilde{\mu}_{x_2(i)}\right)+C_{t6}\left(x_1-\mu_{x_1(i)}\right)\left(\ln x_3-\tilde{\mu}_{x_3(i)}\right)\\&+C_{t7}\left(\ln x_2-\tilde{\mu}_{x_2(i)}\right)\left(\ln x_3-\tilde{\mu}_{x_3(i)}\right)\Big];\end{aligned}$$

and where:

$$C_{t1}=\left[1-\left(\tilde{\rho}_{x_1,x_2(i)}^2+\tilde{\rho}_{x_1,x_3(i)}^2+\tilde{\rho}_{x_2,x_3(i)}^2\right)+2\tilde{\rho}_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_3(i)}\right]^{\frac{1}{2}};$$

$$C_{t2}=\frac{1-\tilde{\rho}_{x_2,x_3(i)}^2}{\sigma_{x_1(i)}^2};\qquad C_{t3}=\frac{1-\tilde{\rho}_{x_1,x_3(i)}^2}{\tilde{\sigma}_{x_2(i)}^2};\qquad C_{t4}=\frac{1-\tilde{\rho}_{x_1,x_2(i)}^2}{\tilde{\sigma}_{x_3(i)}^2};$$

$$C_{t5}=\frac{2\left(\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_3(i)}-\tilde{\rho}_{x_1,x_2(i)}\right)}{\sigma_{x_{1(i)}}\tilde{\sigma}_{x_{2(i)}}};\quad C_{t6}=\frac{2\left(\tilde{\rho}_{x_1,x_2(i)}\tilde{\rho}_{x_2,x_3(i)}-\tilde{\rho}_{x_1,x_3(i)}\right)}{\sigma_{x_{1(i)}}\tilde{\sigma}_{x_{3(i)}}};$$

$$\text{and}\qquad C_{t7}=\frac{2\left(\tilde{\rho}_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_3(i)}-\tilde{\rho}_{x_2,x_3(i)}\right)}{\tilde{\sigma}_{x_{2(i)}}\tilde{\sigma}_{x_{3(i)}}}.$$

In Eq. (S3), $\mu_{x_1(i)}$ is the mean of x_1 in the ith component, $\tilde{\mu}_{x_2(i)}$ is the mean of $\ln x_2$ in the ith component, and $\tilde{\mu}_{x_3(i)}$ is the mean of $\ln x_3$ in the ith component. The ith component standard deviation of x_1 is $\sigma_{x_1(i)}$, the ith component standard deviation of $\ln x_2$ is $\tilde{\sigma}_{x_2(i)}$, and the ith component standard deviation of $\ln x_3$ is $\tilde{\sigma}_{x_3(i)}$. The ith component correlation of x_1 and $\ln x_2$ is $\tilde{\rho}_{x_1,x_2(i)}$, the ith component correlation of x_1 and $\ln x_3$ is $\tilde{\rho}_{x_1,x_3(i)}$, and the ith component correlation of $\ln x_2$ and $\ln x_3$ is $\tilde{\rho}_{x_2,x_3(i)}$.

S3 Functional Form of Bivariate PDFs

There are three types of bivariate PDFs used in the equation set. The first one is a bivariate normal distribution, meaning that the individual marginal for each of x_1 and x_2 is a normal distribution. The functional form of this type of PDF is given by:

$$P_{NN(i)}\left(x_1, x_2\right) = \frac{\exp\left\{-\frac{1}{2}\lambda_{NN}\right\}}{2\pi\sigma_{x_1(i)}\sigma_{x_2(i)}\left(1-\rho^2_{x_1,x_2(i)}\right)^{\frac{1}{2}}}; \quad \text{where} \tag{S4}$$

$$\lambda_{NN} = \frac{1}{1-\rho^2_{x_1,x_2(i)}}\left[\frac{1}{\sigma^2_{x_1(i)}}\left(x_1-\mu_{x_1(i)}\right)^2 + \frac{1}{\sigma^2_{x_2(i)}}\left(x_2-\mu_{x_2(i)}\right)^2 - \frac{2\rho_{x_1,x_2(i)}}{\sigma_{x_1(i)}\sigma_{x_2(i)}}\left(x_1-\mu_{x_1(i)}\right)\left(x_2-\mu_{x_2(i)}\right)\right];$$

where the ith component mean of x_1 is $\mu_{x_1(i)}$, the ith component mean of x_2 is $\mu_{x_2(i)}$, the ith component standard deviation of x_1 is $\sigma_{x_1(i)}$, the ith component standard deviation of x_2 is $\sigma_{x_2(i)}$, and the ith component correlation of x_1 and x_2 is $\rho_{x_1,x_2(i)}$.

The second type of bivariate PDF used in the equation set is a bivariate normal-lognormal distribution, meaning that the individual marginal of x_1 is a normal distribution and the individual marginal of x_2 is a lognormal distribution. The functional form of this type of PDF is given by:

$$P_{NL(i)}\left(x_1, x_2\right) = \frac{\exp\left\{-\frac{1}{2}\lambda_{NL}\right\}}{2\pi\sigma_{x_1(i)}\tilde{\sigma}_{x_2(i)}\left(1-\tilde{\rho}^2_{x_1,x_2(i)}\right)^{\frac{1}{2}} x_2}; \quad \text{where} \tag{S5}$$

$$\lambda_{NL} = \frac{1}{1-\tilde{\rho}^2_{x_1,x_2(i)}}\left[\frac{1}{\sigma^2_{x_1(i)}}\left(x_1-\mu_{x_1(i)}\right)^2 + \frac{1}{\tilde{\sigma}^2_{x_2(i)}}\left(\ln x_2-\tilde{\mu}_{x_2(i)}\right)^2 - \frac{2\tilde{\rho}_{x_1,x_2(i)}}{\sigma_{x_1(i)}\tilde{\sigma}_{x_2(i)}}\left(x_1-\mu_{x_1(i)}\right)\left(\ln x_2-\tilde{\mu}_{x_2(i)}\right)\right];$$

where the ith component mean of x_1 is $\mu_{x_1(i)}$, the ith component mean of $\ln x_2$ is $\tilde{\mu}_{x_2(i)}$, the ith component standard deviation of x_1 is $\sigma_{x_1(i)}$, the ith component standard deviation of

$\ln x_2$ is $\tilde{\sigma}_{x_2(i)}$, and the ith component correlation of x_1 and $\ln x_2$ is $\tilde{\rho}_{x_1,x_2(i)}$.

The third type of bivariate PDF used in the equation set is a bivariate lognormal distribution, meaning that the individual marginal for each of x_1 and x_2 is a lognormal distribution. The functional form of this type of PDF is given by:

$$P_{LL(i)}\left(x_1,x_2\right)=\frac{\exp\left\{-\frac{1}{2}\lambda_{LL}\right\}}{2\pi\tilde{\sigma}_{x_1(i)}\tilde{\sigma}_{x_2(i)}\left(1-\tilde{\rho}^2_{x_1,x_2(i)}\right)^{\frac{1}{2}}x_1x_2};\quad \text{where} \tag{S6}$$

$$\lambda_{LL}=\frac{1}{1-\tilde{\rho}^2_{x_1,x_2(i)}}\left[\frac{1}{\tilde{\sigma}^2_{x_1(i)}}\left(\ln x_1-\tilde{\mu}_{x_1(i)}\right)^2+\frac{1}{\tilde{\sigma}^2_{x_2(i)}}\left(\ln x_2-\tilde{\mu}_{x_2(i)}\right)^2 -\frac{2\tilde{\rho}_{x_1,x_2(i)}}{\tilde{\sigma}_{x_1(i)}\tilde{\sigma}_{x_2(i)}}\left(\ln x_1-\tilde{\mu}_{x_1(i)}\right)\left(\ln x_2-\tilde{\mu}_{x_2(i)}\right)\right];$$

where the ith component mean of $\ln x_1$ is $\tilde{\mu}_{x_1(i)}$, the ith component mean of $\ln x_2$ is $\tilde{\mu}_{x_2(i)}$, the ith component standard deviation of $\ln x_1$ is $\tilde{\sigma}_{x_1(i)}$, the ith component standard deviation of $\ln x_2$ is $\tilde{\sigma}_{x_2(i)}$, and the ith component correlation of $\ln x_1$ and $\ln x_2$ is $\tilde{\rho}_{x_1,x_2(i)}$.

S4 Functional Form of Single-Variable PDFs

There are two types of single-variable (univariate) PDFs used in the equation set. The first one is a normal distribution. The functional form of this type of PDF is given by:

$$P_{N(i)}\left(x\right)=\frac{1}{\left(2\pi\right)^{\frac{1}{2}}\sigma_{x(i)}}\exp\left\{\frac{-\left(x-\mu_{x(i)}\right)^2}{2\,\sigma^2_{x(i)}}\right\}; \tag{S7}$$

where the ith component mean of x is $\mu_{x(i)}$ and the ith component standard deviation of x is $\sigma_{x(i)}$. The second type of univariate PDF used in this equation set is a lognormal distribution. If the natural logarithm was taken for every point in a lognormal distribution, the resulting distribution would be a normal distribution. The functional form of this type of PDF is given by:

$$P_{L(i)}\left(x\right) = \frac{1}{\left(2\pi\right)^{\frac{1}{2}}\,\tilde{\sigma}_{x(i)}\;x}\exp\left\{\frac{-\left(\ln x - \tilde{\mu}_{x(i)}\right)^{2}}{2\;\tilde{\sigma}^{2}_{x(i)}}\right\}; \tag{S8}$$

where the ith component mean of $\ln x$ is $\tilde{\mu}_{x(i)}$ and the ith component standard deviation of $\ln x$ is $\tilde{\sigma}_{x(i)}$.

S5 Quadrivariate PDF Integrals of Covariance Form

The integrals of the general form

$$G_{QC} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}\Big(x_1 - C_1\Big)\Big(x_2^{\alpha}\left(H\left(-x_2\right)\right)^{\alpha} x_3^{\beta} x_4^{\gamma} - C_2\Big)$$
$$\times\, P_{NNLL}\left(x_1, x_2, x_3, x_4\right) \mathrm{d}x_4\, \mathrm{d}x_3\, \mathrm{d}x_2\, \mathrm{d}x_1$$

are referred to as quadrivariate PDF integrals of covariance form. Both C_1 and C_2 are constants, and when they both represent the appropriate overall mean values, the resulting integral is a covariance. The quadrivariate PDF, $P_{NNLL}\left(x_1, x_2, x_3, x_4\right)$, is a normal-normal-lognormal-lognormal PDF, meaning that the individual marginals of both x_1 and x_2 are normal distributions and the individual marginals of both x_3 and x_4 are lognormal distributions. The Heaviside step function is denoted $H\left(x\right)$. The above integral has 16 sub-forms. When one or more of the variables is constant (has a standard deviation of 0), the integral simplifies and reduces.

In the solutions below, denoted G_{QC}, μ_{x_1} and σ_{x_1} denote the mean and standard deviation of x_1 in the quadrivariate PDF, μ_{x_2} and σ_{x_2} denote the mean and standard deviation of x_2 in the quadrivariate PDF, μ_{x_3} and σ_{x_3} denote the mean and standard deviation of x_3 in the quadrivariate PDF, and μ_{x_4} and σ_{x_4} denote the mean and standard deviation of x_4 in the quadrivariate PDF. For lognormal variates, $\tilde{\mu}_{x_3}$ and $\tilde{\sigma}_{x_3}$ denote the mean and standard deviation of $\ln x_3$ in the quadrivariate PDF, while $\tilde{\mu}_{x_4}$ and $\tilde{\sigma}_{x_4}$ denote the mean and standard deviation of $\ln x_4$ in the quadrivariate PDF. The correlation of x_1 and x_2 is denoted ρ_{x_1,x_2}, the correlation of x_1 and $\ln x_3$ is denoted $\tilde{\rho}_{x_1,x_3}$, the correlation of x_1 and $\ln x_4$ is denoted $\tilde{\rho}_{x_1,x_4}$, the correlation of x_2 and $\ln x_3$ is denoted $\tilde{\rho}_{x_2,x_3}$, the correlation of x_2 and $\ln x_4$ is denoted $\tilde{\rho}_{x_2,x_4}$, and the correlation of $\ln x_3$ and $\ln x_4$ is denoted $\tilde{\rho}_{x_3,x_4}$. The gamma function is denoted $\Gamma\left(x\right)$ and the parabolic cylinder function of order ν is denoted $D_{\nu}\left(x\right)$.

When x_1, x_2, x_3, and x_4 all vary ($\sigma_{x_1} > 0$, $\sigma_{x_2} > 0$, $\sigma_{x_3} > 0$, and $\sigma_{x_4} > 0$), the solution is

$$
\begin{aligned}
G_{QC} = & \frac{1}{\sqrt{2\pi}} \left(-\sigma_{x_2}\right)^{\alpha} \exp\left\{ \tilde{\mu}_{x_3}\beta + \tilde{\mu}_{x_4}\gamma + \frac{1}{2}\left(1-\tilde{\rho}^2_{x_2,x_3}\right)\tilde{\sigma}^2_{x_3}\beta^2 \right. \\
& \left. + \frac{1}{2}\left(1-\tilde{\rho}^2_{x_2,x_4}\right)\tilde{\sigma}^2_{x_4}\gamma^2 + \left(\tilde{\rho}_{x_3,x_4} - \tilde{\rho}_{x_2,x_3}\tilde{\rho}_{x_2,x_4}\right)\tilde{\sigma}_{x_3}\beta\tilde{\sigma}_{x_4}\gamma \right\} \\
& \times \exp\left\{ \frac{1}{4}\varsigma^2 - \frac{\mu_{x_2}}{\sigma_{x_2}}\varsigma + \frac{1}{2}\frac{\mu^2_{x_2}}{\sigma^2_{x_2}} \right\} \\
& \times \Bigg(- \rho_{x_1,x_2}\sigma_{x_1}\Gamma\left(\alpha+2\right) D_{-(\alpha+2)}\left(\varsigma\right) \\
& \quad + \bigg(\mu_{x_1} - C_1 - \frac{\mu_{x_2}}{\sigma_{x_2}}\rho_{x_1,x_2}\sigma_{x_1} + \left(\tilde{\rho}_{x_1,x_3} - \rho_{x_1,x_2}\tilde{\rho}_{x_2,x_3}\right)\sigma_{x_1}\tilde{\sigma}_{x_3}\beta \\
& \quad\quad + \left(\tilde{\rho}_{x_1,x_4} - \rho_{x_1,x_2}\tilde{\rho}_{x_2,x_4}\right)\sigma_{x_1}\tilde{\sigma}_{x_4}\gamma \bigg) \Gamma\left(\alpha+1\right) D_{-(\alpha+1)}\left(\varsigma\right) \Bigg) \\
& - C_2\left(\mu_{x_1} - C_1\right);
\end{aligned}
\tag{S9}
$$

where $\quad \varsigma = \dfrac{\mu_{x_2}}{\sigma_{x_2}} + \tilde{\rho}_{x_2,x_3}\tilde{\sigma}_{x_3}\beta + \tilde{\rho}_{x_2,x_4}\tilde{\sigma}_{x_4}\gamma.$

There are four sub-forms that contain one constant variable. When x_1 is constant, but x_2, x_3, and x_4 vary, the solution is

$$
\begin{aligned}
G_{QC} = & \frac{1}{\sqrt{2\pi}} \left(\mu_{x_1} - C_1\right)\left(-\sigma_{x_2}\right)^{\alpha} \\
& \times \exp\left\{ \tilde{\mu}_{x_3}\beta + \tilde{\mu}_{x_4}\gamma + \frac{1}{2}\left(1-\tilde{\rho}^2_{x_2,x_3}\right)\tilde{\sigma}^2_{x_3}\beta^2 \right. \\
& \left. \quad + \frac{1}{2}\left(1-\tilde{\rho}^2_{x_2,x_4}\right)\tilde{\sigma}^2_{x_4}\gamma^2 + \left(\tilde{\rho}_{x_3,x_4} - \tilde{\rho}_{x_2,x_3}\tilde{\rho}_{x_2,x_4}\right)\tilde{\sigma}_{x_3}\beta\tilde{\sigma}_{x_4}\gamma \right\} \\
& \times \exp\left\{ \frac{1}{4}\varsigma^2 - \frac{\mu_{x_2}}{\sigma_{x_2}}\varsigma + \frac{1}{2}\frac{\mu^2_{x_2}}{\sigma^2_{x_2}} \right\} \Gamma\left(\alpha+1\right) D_{-(\alpha+1)}\left(\varsigma\right) \\
& - C_2\left(\mu_{x_1} - C_1\right);
\end{aligned}
\tag{S10}
$$

where $\quad \varsigma = \dfrac{\mu_{x_2}}{\sigma_{x_2}} + \tilde{\rho}_{x_2,x_3}\tilde{\sigma}_{x_3}\beta + \tilde{\rho}_{x_2,x_4}\tilde{\sigma}_{x_4}\gamma.$

When x_2 is constant, but x_1, x_3, and x_4 vary, the solution is

$$G_{QC} = \begin{cases} \mu_{x_2}^{\alpha}\Big(\mu_{x_1} - C_1 + \tilde{\rho}_{x_1,x_3}\sigma_{x_1}\tilde{\sigma}_{x_3}\beta + \tilde{\rho}_{x_1,x_4}\sigma_{x_1}\tilde{\sigma}_{x_4}\gamma\Big) \\ \times \exp\left\{\tilde{\mu}_{x_3}\beta + \tilde{\mu}_{x_4}\gamma + \frac{1}{2}\tilde{\sigma}_{x_3}^2\beta^2 + \frac{1}{2}\tilde{\sigma}_{x_4}^2\gamma^2 + \tilde{\rho}_{x_3,x_4}\tilde{\sigma}_{x_3}\beta\tilde{\sigma}_{x_4}\gamma\right\} \\ -C_2\left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} \leq 0;\ \text{and} \\ \\ -C_2\left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} > 0. \end{cases} \tag{S11}$$

When x_3 is constant, but x_1, x_2, and x_4 vary, the solution is

$$\begin{aligned} G_{QC} = {} & \frac{1}{\sqrt{2\pi}}\left(-\sigma_{x_2}\right)^{\alpha}\mu_{x_3}^{\beta}\exp\left\{\tilde{\mu}_{x_4}\gamma + \frac{1}{2}\tilde{\sigma}_{x_4}^2\gamma^2 - \frac{1}{4}\varsigma^2\right\} \\ & \times \Bigg(- \rho_{x_1,x_2}\sigma_{x_1}\Gamma\left(\alpha+2\right)D_{-(\alpha+2)}\left(\varsigma\right) \\ & \quad + \left(\mu_{x_1} - C_1 - \frac{\mu_{x_2}}{\sigma_{x_2}}\rho_{x_1,x_2}\sigma_{x_1} + \left(\tilde{\rho}_{x_1,x_4} - \rho_{x_1,x_2}\tilde{\rho}_{x_2,x_4}\right)\sigma_{x_1}\tilde{\sigma}_{x_4}\gamma\right) \\ & \qquad \times \Gamma\left(\alpha+1\right)D_{-(\alpha+1)}\left(\varsigma\right)\Bigg) \\ & - C_2\left(\mu_{x_1} - C_1\right); \end{aligned} \tag{S12}$$

where $\quad \varsigma = \dfrac{\mu_{x_2}}{\sigma_{x_2}} + \tilde{\rho}_{x_2,x_4}\tilde{\sigma}_{x_4}\gamma.$

When x_4 is constant, but x_1, x_2, and x_3 vary, the solution is

$$\begin{aligned}G_{QC} = &\frac{1}{\sqrt{2\pi}} \left(-\sigma_{x_2}\right)^{\alpha} \mu_{x_4}^{\gamma} \exp\left\{\tilde{\mu}_{x_3}\beta + \frac{1}{2}\tilde{\sigma}_{x_3}^2\beta^2 - \frac{1}{4}\varsigma^2\right\} \\ &\times \Bigg(- \rho_{x_1,x_2}\sigma_{x_1}\Gamma\left(\alpha+2\right) D_{-(\alpha+2)}\left(\varsigma\right) \\ &\quad + \left(\mu_{x_1} - C_1 - \frac{\mu_{x_2}}{\sigma_{x_2}}\rho_{x_1,x_2}\sigma_{x_1} + \left(\tilde{\rho}_{x_1,x_3} - \rho_{x_1,x_2}\tilde{\rho}_{x_2,x_3}\right)\sigma_{x_1}\tilde{\sigma}_{x_3}\beta\right) \\ &\qquad \times \Gamma\left(\alpha+1\right) D_{-(\alpha+1)}\left(\varsigma\right) \Bigg) \\ &- C_2\left(\mu_{x_1} - C_1\right);\end{aligned} \tag{S13}$$

where $\varsigma = \dfrac{\mu_{x_2}}{\sigma_{x_2}} + \tilde{\rho}_{x_2,x_3}\tilde{\sigma}_{x_3}\beta.$

There are six sub-forms that contain two constant variables. When both x_1 and x_2 are constant, but both x_3 and x_4 vary, the solution is

$$G_{QC} = \begin{cases} \left(\mu_{x_1} - C_1\right)\mu_{x_2}^{\alpha} \exp\left\{\tilde{\mu}_{x_3}\beta + \tilde{\mu}_{x_4}\gamma + \frac{1}{2}\tilde{\sigma}_{x_3}^2\beta^2 + \frac{1}{2}\tilde{\sigma}_{x_4}^2\gamma^2 + \tilde{\rho}_{x_3,x_4}\tilde{\sigma}_{x_3}\beta\tilde{\sigma}_{x_4}\gamma\right\} & \\ -C_2\left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} \leq 0;\ \text{and} & \\ & \\ -C_2\left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} > 0. & \end{cases} \tag{S14}$$

When both x_1 and x_3 are constant, but both x_2 and x_4 vary, the solution is

$$\begin{aligned}G_{QC} = &\frac{1}{\sqrt{2\pi}} \left(\mu_{x_1} - C_1\right)\left(-\sigma_{x_2}\right)^{\alpha} \mu_{x_3}^{\beta} \exp\left\{\tilde{\mu}_{x_4}\gamma + \frac{1}{2}\tilde{\sigma}_{x_4}^2\gamma^2 - \frac{1}{4}\varsigma^2\right\}\Gamma\left(\alpha+1\right) D_{-(\alpha+1)}\left(\varsigma\right) \\ &- C_2\left(\mu_{x_1} - C_1\right);\end{aligned} \tag{S15}$$

where $\varsigma = \dfrac{\mu_{x_2}}{\sigma_{x_2}} + \tilde{\rho}_{x_2,x_4}\tilde{\sigma}_{x_4}\gamma.$

When both x_1 and x_4 are constant, but both x_2 and x_3 vary, the solution is

$$G_{QC} = \frac{1}{\sqrt{2\pi}} \left(\mu_{x_1} - C_1\right) \left(-\sigma_{x_2}\right)^{\alpha} \mu_{x_4}^{\gamma} \exp\left\{\tilde{\mu}_{x_3}\beta + \frac{1}{2}\tilde{\sigma}_{x_3}^2\beta^2 - \frac{1}{4}\varsigma^2\right\} \Gamma\left(\alpha+1\right) D_{-(\alpha+1)}\left(\varsigma\right) - C_2\left(\mu_{x_1} - C_1\right); \tag{S16}$$

where $\varsigma = \frac{\mu_{x_2}}{\sigma_{x_2}} + \tilde{\rho}_{x_2,x_3}\tilde{\sigma}_{x_3}\beta.$

When both x_2 and x_3 are constant, but both x_1 and x_4 vary, the solution is

$$G_{QC} = \begin{cases} \mu_{x_2}^{\alpha}\mu_{x_3}^{\beta}\left(\mu_{x_1} - C_1 + \tilde{\rho}_{x_1,x_4}\sigma_{x_1}\tilde{\sigma}_{x_4}\gamma\right) \exp\left\{\tilde{\mu}_{x_4}\gamma + \frac{1}{2}\tilde{\sigma}_{x_4}^2\gamma^2\right\} \\ -C_2\left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} \le 0; \text{ and} \\ \\ -C_2\left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} > 0. \end{cases} \tag{S17}$$

When both x_2 and x_4 are constant, but both x_1 and x_3 vary, the solution is

$$G_{QC} = \begin{cases} \mu_{x_2}^{\alpha}\mu_{x_4}^{\gamma}\left(\mu_{x_1} - C_1 + \tilde{\rho}_{x_1,x_3}\sigma_{x_1}\tilde{\sigma}_{x_3}\beta\right) \exp\left\{\tilde{\mu}_{x_3}\beta + \frac{1}{2}\tilde{\sigma}_{x_3}^2\beta^2\right\} \\ -C_2\left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} \le 0; \text{ and} \\ \\ -C_2\left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} > 0. \end{cases} \tag{S18}$$

When both x_3 and x_4 are constant, but both x_1 and x_2 vary, the solution is

$$\begin{aligned} G_{QC} = & \frac{1}{\sqrt{2\pi}} \left(-\sigma_{x_2}\right)^{\alpha} \mu_{x_3}^{\beta} \mu_{x_4}^{\gamma} \exp\left\{ -\frac{1}{4}\frac{\mu_{x_2}^2}{\sigma_{x_2}^2} \right\} \\ & \times \Bigg(-\rho_{x_1,x_2}\sigma_{x_1}\Gamma\left(\alpha+2\right) D_{-(\alpha+2)}\left(\frac{\mu_{x_2}}{\sigma_{x_2}}\right) \\ & \quad + \left(\mu_{x_1} - C_1 - \frac{\mu_{x_2}}{\sigma_{x_2}}\rho_{x_1,x_2}\sigma_{x_1}\right)\Gamma\left(\alpha+1\right) D_{-(\alpha+1)}\left(\frac{\mu_{x_2}}{\sigma_{x_2}}\right)\Bigg) \\ & - C_2\left(\mu_{x_1} - C_1\right). \end{aligned} \tag{S19}$$

There are four sub-forms that contain three constant variables. When x_1, x_2, and x_3 are constant, but x_4 varies, the solution is

$$G_{QC} = \begin{cases} \left(\mu_{x_1} - C_1\right) \mu_{x_2}^{\alpha} \mu_{x_3}^{\beta} \exp\left\{\tilde{\mu}_{x_4}\gamma + \frac{1}{2}\tilde{\sigma}_{x_4}^2\gamma^2\right\} & \\ -C_2\left(\mu_{x_1} - C_1\right), & \text{when } \mu_{x_2} \leq 0;\ \text{and} \\ -C_2\left(\mu_{x_1} - C_1\right), & \text{when } \mu_{x_2} > 0. \end{cases} \tag{S20}$$

When x_1, x_2, and x_4 are constant, but x_2 varies, the solution is

$$G_{QC} = \begin{cases} \left(\mu_{x_1} - C_1\right) \mu_{x_2}^{\alpha} \mu_{x_4}^{\gamma} \exp\left\{\tilde{\mu}_{x_3}\beta + \frac{1}{2}\tilde{\sigma}_{x_3}^2\beta^2\right\} & \\ -C_2\left(\mu_{x_1} - C_1\right), & \text{when } \mu_{x_2} \leq 0;\ \text{and} \\ -C_2\left(\mu_{x_1} - C_1\right), & \text{when } \mu_{x_2} > 0. \end{cases} \tag{S21}$$

When x_1, x_3, and x_4 are constant, but x_2 varies, the solution is

$$\begin{aligned} G_{QC} = & \frac{1}{\sqrt{2\pi}} \left(\mu_{x_1} - C_1\right)\left(-\sigma_{x_2}\right)^{\alpha} \mu_{x_3}^{\beta} \mu_{x_4}^{\gamma} \exp\left\{ -\frac{1}{4}\frac{\mu_{x_2}^2}{\sigma_{x_2}^2} \right\}\Gamma\left(\alpha+1\right) D_{-(\alpha+1)}\left(\frac{\mu_{x_2}}{\sigma_{x_2}}\right) \\ & - C_2\left(\mu_{x_1} - C_1\right). \end{aligned} \tag{S22}$$

When x_2, x_3, and x_4 are constant, but x_1 varies, the solution is

$$G_{QC} = \begin{cases} \left(\mu_{x_1} - C_1\right)\left(\mu_{x_2}^{\alpha}\mu_{x_3}^{\beta}\mu_{x_4}^{\gamma} - C_2\right), & \text{when } \mu_{x_2} \leq 0;\ \text{and} \\ -C_2\left(\mu_{x_1} - C_1\right), & \text{when } \mu_{x_2} > 0. \end{cases} \tag{S23}$$

When x_1, x_2, x_3, and x_4 are all constant ($\sigma_{x_1} = 0$, $\sigma_{x_2} = 0$, $\sigma_{x_3} = 0$, and $\sigma_{x_4} = 0$), the solution is

$$G_{QC} = \begin{cases} \left(\mu_{x_1} - C_1\right)\left(\mu_{x_2}^{\alpha}\mu_{x_3}^{\beta}\mu_{x_4}^{\gamma} - C_2\right), & \text{when } \mu_{x_2} \leq 0;\ \text{and} \\ -C_2\left(\mu_{x_1} - C_1\right), & \text{when } \mu_{x_2} > 0. \end{cases} \tag{S24}$$

S6 Trivariate PDF Integrals of Covariance Form

The integrals of the general form

$$G_{TC} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\Big(x_1 - C_1\Big)\Big(x_2^{\alpha}\left(H\left(x_2\right)\right)^{\alpha} x_3^{\beta} - C_2\Big) P_{NNL}\left(x_1, x_2, x_3\right) \mathrm{d}x_3 \, \mathrm{d}x_2 \, \mathrm{d}x_1$$

are referred to as trivariate PDF integrals of covariance form. Both C_1 and C_2 are constants, and when they both represent the appropriate overall mean values, the resulting integral is a covariance. The trivariate PDF, $P_{NNL}\left(x_1, x_2, x_3\right)$, is a normal-normal-lognormal PDF, meaning that the individual marginals of both x_1 and x_2 are normal distributions and the individual marginal of x_3 is a lognormal distribution. The above integral has eight sub-forms. When one or more of the variables is constant (has a standard deviation of 0), the integral simplifies and reduces.

When x_1, x_2, and x_3 all vary ($\sigma_{x_1} > 0$, $\sigma_{x_2} > 0$, and $\sigma_{x_3} > 0$), the solution, denoted G_{TC}, is

$$
\begin{aligned}
G_{TC} = & \frac{1}{\sqrt{2\pi}}\,\sigma_{x_2}^{\alpha}\,\exp\left\{\tilde{\mu}_{x_3}\beta + \frac{1}{2}\tilde{\sigma}_{x_3}^2\beta^2 - \frac{1}{4}\varsigma^2\right\} \\
& \times\Bigg(\rho_{x_1,x_2}\sigma_{x_1}\Gamma\left(\alpha+2\right)D_{-(\alpha+2)}\left(-\varsigma\right) \\
& \quad + \left(\mu_{x_1} - C_1 - \frac{\mu_{x_2}}{\sigma_{x_2}}\rho_{x_1,x_2}\sigma_{x_1} + \left(\tilde{\rho}_{x_1,x_3} - \rho_{x_1,x_2}\tilde{\rho}_{x_2,x_3}\right)\sigma_{x_1}\tilde{\sigma}_{x_3}\beta\right) \\
& \quad\quad \times\Gamma\left(\alpha+1\right)D_{-(\alpha+1)}\left(-\varsigma\right)\Bigg) \\
& - C_2\left(\mu_{x_1} - C_1\right);
\end{aligned}
\tag{S25}
$$

where $\quad \varsigma = \dfrac{\mu_{x_2}}{\sigma_{x_2}} + \tilde{\rho}_{x_2,x_3}\tilde{\sigma}_{x_3}\beta.$

There are three sub-forms that contain one constant variable. When x_1 is constant, but x_2 and x_3 vary, the solution is

$$
\begin{aligned}
G_{TC} = & \frac{1}{\sqrt{2\pi}}\left(\mu_{x_1} - C_1\right)\sigma_{x_2}^{\alpha}\,\exp\left\{\tilde{\mu}_{x_3}\beta + \frac{1}{2}\tilde{\sigma}_{x_3}^2\beta^2 - \frac{1}{4}\varsigma^2\right\}\Gamma\left(\alpha+1\right)D_{-(\alpha+1)}\left(-\varsigma\right) \\
& - C_2\left(\mu_{x_1} - C_1\right);
\end{aligned}
\tag{S26}
$$

where $\quad \varsigma = \dfrac{\mu_{x_2}}{\sigma_{x_2}} + \tilde{\rho}_{x_2,x_3}\tilde{\sigma}_{x_3}\beta.$

When x_2 is constant, but x_1 and x_3 vary, the solution is

$$
G_{TC} = \begin{cases}
\mu_{x_2}^{\alpha}\left(\mu_{x_1} - C_1 + \tilde{\rho}_{x_1,x_3}\sigma_{x_1}\tilde{\sigma}_{x_3}\beta\right)\exp\left\{\tilde{\mu}_{x_3}\beta + \frac{1}{2}\tilde{\sigma}_{x_3}^2\beta^2\right\} & \\
-C_2\left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} \geq 0; \text{ and} & \\
 & \\
-C_2\left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} < 0. &
\end{cases}
\tag{S27}
$$

When x_3 is constant, but x_1 and x_2 vary, the solution is

$$\begin{aligned} G_{TC} = & \frac{1}{\sqrt{2\pi}} \sigma_{x_2}^{\alpha} \mu_{x_3}^{\beta} \exp\left\{ -\frac{1}{4} \frac{\mu_{x_2}^2}{\sigma_{x_2}^2} \right\} \\ & \times \Bigg(\rho_{x_1,x_2} \sigma_{x_1} \Gamma(\alpha+2) D_{-(\alpha+2)} \left(-\frac{\mu_{x_2}}{\sigma_{x_2}} \right) \\ & \quad + \left(\mu_{x_1} - C_1 - \frac{\mu_{x_2}}{\sigma_{x_2}} \rho_{x_1,x_2} \sigma_{x_1} \right) \Gamma(\alpha+1) D_{-(\alpha+1)} \left(-\frac{\mu_{x_2}}{\sigma_{x_2}} \right) \Bigg) \\ & - C_2 (\mu_{x_1} - C_1). \end{aligned} \tag{S28}$$

There are three sub-forms that contain two constant variables. When both x_1 and x_2 are constant, but x_3 varies, the solution is

$$G_{TC} = \begin{cases} \mu_{x_2}^{\alpha} (\mu_{x_1} - C_1) \exp\left\{ \tilde{\mu}_{x_3} \beta + \frac{1}{2} \tilde{\sigma}_{x_3}^2 \beta^2 \right\} - C_2 (\mu_{x_1} - C_1), \text{ when } \mu_{x_2} \geq 0; \text{ and} \\ -C_2 (\mu_{x_1} - C_1), \qquad \text{when } \mu_{x_2} < 0. \end{cases} \tag{S29}$$

When both x_1 and x_3 are constant, but x_2 varies, the solution is

$$\begin{aligned} G_{TC} = & \frac{1}{\sqrt{2\pi}} (\mu_{x_1} - C_1) \sigma_{x_2}^{\alpha} \mu_{x_3}^{\beta} \exp\left\{ -\frac{1}{4} \frac{\mu_{x_2}^2}{\sigma_{x_2}^2} \right\} \Gamma(\alpha+1) D_{-(\alpha+1)} \left(-\frac{\mu_{x_2}}{\sigma_{x_2}} \right) \\ & - C_2 (\mu_{x_1} - C_1). \end{aligned} \tag{S30}$$

When both x_2 and x_3 are constant, but x_1 varies, the solution is

$$G_{TC} = \begin{cases} \big(\mu_{x_1} - C_1\big)\big(\mu_{x_2}^{\alpha} \mu_{x_3}^{\beta} - C_2\big), \qquad \text{when } \mu_{x_2} \geq 0; \text{ and} \\ -C_2 (\mu_{x_1} - C_1), \qquad \text{when } \mu_{x_2} < 0. \end{cases} \tag{S31}$$

When x_1, x_2, and x_3 are all constant ($\sigma_{x_1} = 0$, $\sigma_{x_2} = 0$, and $\sigma_{x_3} = 0$), the solution is

$$G_{TC} = \begin{cases} \left(\mu_{x_1} - C_1\right)\left(\mu_{x_2}^{\alpha}\mu_{x_3}^{\beta} - C_2\right), \qquad \text{when } \mu_{x_2} \geq 0;\ \text{and} \\ -C_2\left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} < 0. \end{cases} \tag{S32}$$

S7 Trivariate PDF Integrals of Mean Form

The integrals of the general form

$$\begin{aligned} G_{TM} &= \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} x_1^{\alpha}\left(H\left(-x_1\right)\right)^{\alpha} x_2^{\beta}x_3^{\gamma}P_{NLL}\left(x_1,x_2,x_3\right)\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1 \\ &= \int_{-\infty}^{0}\int_{0}^{\infty}\int_{0}^{\infty} x_1^{\alpha}x_2^{\beta}x_3^{\gamma}P_{NLL}\left(x_1,x_2,x_3\right)\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1 \end{aligned}$$

are referred to as trivariate PDF integrals of mean form. The trivariate PDF, $P_{NLL}\left(x_1,x_2,x_3\right)$, is a normal-lognormal-lognormal PDF, meaning that the individual marginal of x_1 is a normal distribution and the individual marginals of both x_2 and x_3 are lognormal distributions. The above integral has eight sub-forms. When one or more of the variables is constant (has a standard deviation of 0), the integral simplifies and reduces.

In the solutions below, denoted G_{TM}, μ_{x_1} and σ_{x_1} denote the mean and standard deviation of x_1 in the trivariate PDF, μ_{x_2} and σ_{x_2} denote the mean and standard deviation of x_2 in the trivariate PDF, and μ_{x_3} and σ_{x_3} denote the mean and standard deviation of x_3 in the trivariate PDF. For lognormal variates, $\tilde{\mu}_{x_2}$ and $\tilde{\sigma}_{x_2}$ denote the mean and standard deviation of $\ln x_2$ in the trivariate PDF, while $\tilde{\mu}_{x_3}$ and $\tilde{\sigma}_{x_3}$ denote the mean and standard deviation of $\ln x_3$ in the trivariate PDF. The correlation of x_1 and $\ln x_2$ is denoted $\tilde{\rho}_{x_1,x_2}$, the correlation of x_1 and $\ln x_3$ is denoted $\tilde{\rho}_{x_1,x_3}$, and the correlation of $\ln x_2$ and $\ln x_3$ is denoted $\tilde{\rho}_{x_2,x_3}$. The gamma function is denoted $\Gamma\left(x\right)$ and the parabolic cylinder function of order ν is denoted $D_{\nu}\left(x\right)$.

When x_1, x_2, and x_3 all vary ($\sigma_{x_1} > 0$, $\sigma_{x_2} > 0$, and $\sigma_{x_3} > 0$), the solution is

$$G_{TM} = \frac{1}{\sqrt{2\pi}} \left(-\sigma_{x_1}\right)^{\alpha} \exp\left\{ \tilde{\mu}_{x_2}\beta + \tilde{\mu}_{x_3}\gamma + \frac{1}{2}\left(1 - \tilde{\rho}^2_{x_1,x_2}\right)\tilde{\sigma}^2_{x_2}\beta^2 + \frac{1}{2}\left(1 - \tilde{\rho}^2_{x_1,x_3}\right)\tilde{\sigma}^2_{x_3}\gamma^2 + \left(\tilde{\rho}_{x_2,x_3} - \tilde{\rho}_{x_1,x_2}\tilde{\rho}_{x_1,x_3}\right)\tilde{\sigma}_{x_2}\beta\tilde{\sigma}_{x_3}\gamma \right\} \times \exp\left\{ \frac{1}{4}\varsigma^2 - \frac{\mu_{x_1}}{\sigma_{x_1}}\varsigma + \frac{1}{2}\frac{\mu^2_{x_1}}{\sigma^2_{x_1}} \right\} \Gamma\left(\alpha+1\right) D_{-(\alpha+1)}\left(\varsigma\right); \tag{S33}$$

where $\varsigma = \dfrac{\mu_{x_1}}{\sigma_{x_1}} + \tilde{\rho}_{x_1,x_2}\tilde{\sigma}_{x_2}\beta + \tilde{\rho}_{x_1,x_3}\tilde{\sigma}_{x_3}\gamma.$

There are three sub-forms that contain one constant variable. When x_1 is constant, but x_2 and x_3 vary, the solution is

$$G_{TM} = \begin{cases} \mu^{\alpha}_{x_1} \exp\left\{ \tilde{\mu}_{x_2}\beta + \tilde{\mu}_{x_3}\gamma + \frac{1}{2}\tilde{\sigma}^2_{x_2}\beta^2 + \frac{1}{2}\tilde{\sigma}^2_{x_3}\gamma^2 + \tilde{\rho}_{x_2,x_3}\tilde{\sigma}_{x_2}\beta\tilde{\sigma}_{x_3}\gamma \right\}, \\ \quad \text{when } \mu_{x_1} \le 0; \text{ and} \\ 0, \text{ when } \mu_{x_1} > 0. \end{cases} \tag{S34}$$

When x_2 is constant, but x_1 and x_3 vary, the solution is

$$G_{TM} = \frac{1}{\sqrt{2\pi}} \left(-\sigma_{x_1}\right)^{\alpha} \mu^{\beta}_{x_2} \exp\left\{ \tilde{\mu}_{x_3}\gamma + \frac{1}{2}\tilde{\sigma}^2_{x_3}\gamma^2 - \frac{1}{4}\varsigma^2 \right\} \Gamma\left(\alpha+1\right) D_{-(\alpha+1)}\left(\varsigma\right); \tag{S35}$$

where $\varsigma = \dfrac{\mu_{x_1}}{\sigma_{x_1}} + \tilde{\rho}_{x_1,x_3}\tilde{\sigma}_{x_3}\gamma.$

When x_3 is constant, but x_1 and x_2 vary, the solution is

$$G_{TM} = \frac{1}{\sqrt{2\pi}} \left(-\sigma_{x_1}\right)^{\alpha} \mu^{\gamma}_{x_3} \exp\left\{ \tilde{\mu}_{x_2}\beta + \frac{1}{2}\tilde{\sigma}^2_{x_2}\beta^2 - \frac{1}{4}\varsigma^2 \right\} \Gamma\left(\alpha+1\right) D_{-(\alpha+1)}\left(\varsigma\right); \tag{S36}$$

where $\varsigma = \dfrac{\mu_{x_1}}{\sigma_{x_1}} + \tilde{\rho}_{x_1,x_2}\tilde{\sigma}_{x_2}\beta.$

There are three sub-forms that contain two constant variables. When both x_1 and x_2 are constant, but x_3 varies, the solution is

$$G_{TM} = \begin{cases} \mu_{x_1}^{\alpha}\mu_{x_2}^{\beta}\exp\left\{\tilde{\mu}_{x_3}\gamma + \frac{1}{2}\tilde{\sigma}_{x_3}^2\gamma^2\right\}, \text{ when } \mu_{x_1} \leq 0;\ \text{and} \\ 0,\ \text{when } \mu_{x_1} > 0. \end{cases} \tag{S37}$$

When both x_1 and x_3 are constant, but x_2 varies, the solution is

$$G_{TM} = \begin{cases} \mu_{x_1}^{\alpha}\mu_{x_3}^{\gamma}\exp\left\{\tilde{\mu}_{x_2}\beta + \frac{1}{2}\tilde{\sigma}_{x_2}^2\beta^2\right\}, \text{ when } \mu_{x_1} \leq 0;\ \text{and} \\ 0,\ \text{when } \mu_{x_1} > 0. \end{cases} \tag{S38}$$

When both x_2 and x_3 are constant, but x_1 varies, the solution is

$$G_{TM} = \frac{1}{\sqrt{2\pi}}\left(-\sigma_{x_1}\right)^{\alpha}\mu_{x_2}^{\beta}\mu_{x_3}^{\gamma}\exp\left\{-\frac{1}{4}\frac{\mu_{x_1}^2}{\sigma_{x_1}^2}\right\}\Gamma\left(\alpha+1\right)D_{-(\alpha+1)}\left(\frac{\mu_{x_1}}{\sigma_{x_1}}\right). \tag{S39}$$

When x_1, x_2, and x_3 are all constant ($\sigma_{x_1} = 0$, $\sigma_{x_2} = 0$, and $\sigma_{x_3} = 0$), the solution is

$$G_{TM} = \begin{cases} \mu_{x_1}^{\alpha}\mu_{x_2}^{\beta}\mu_{x_3}^{\gamma}, \text{ when } \mu_{x_1} \leq 0;\ \text{and} \\ 0,\ \text{when } \mu_{x_1} > 0. \end{cases} \tag{S40}$$

S8 Bivariate PDF Integrals of Mean Form

The integrals of the general form

$$\begin{aligned} G_{BM} &= \int_{-\infty}^{\infty}\int_{0}^{\infty} x_1^{\alpha}\left(H\left(x_1\right)\right)^{\alpha}x_2^{\beta}P_{NL}\left(x_1,x_2\right)\mathrm{d}x_2\,\mathrm{d}x_1 \\ &= \int_{0}^{\infty}\int_{0}^{\infty} x_1^{\alpha}x_2^{\beta}P_{NL}\left(x_1,x_2\right)\mathrm{d}x_2\,\mathrm{d}x_1 \end{aligned}$$

are referred to as bivariate PDF integrals of mean form. The bivariate PDF, $P_{NL}(x_1, x_2)$, is a normal-lognormal PDF, meaning that the individual marginal of x_1 is a normal distribution and the individual marginal of x_2 is a lognormal distribution. The above integral has four sub-forms. When one or more of the variables is constant (has a standard deviation of 0), the integral simplifies and reduces. In the solutions below, denoted G_{BM}, the notation is the same as in Section S7.

When both x_1 and x_2 vary ($\sigma_{x_1} > 0$ and $\sigma_{x_2} > 0$), the solution is

$$G_{BM} = \frac{1}{\sqrt{2\pi}} \sigma_{x_1}^{\alpha} \exp\left\{ \tilde{\mu}_{x_2}\beta + \frac{1}{2}\tilde{\sigma}_{x_2}^2\beta^2 - \frac{1}{4}\varsigma^2 \right\} \Gamma(\alpha+1) D_{-(\alpha+1)}(-\varsigma); \tag{S41}$$

where $\varsigma = \dfrac{\mu_{x_1}}{\sigma_{x_1}} + \tilde{\rho}_{x_1,x_2}\tilde{\sigma}_{x_2}\beta.$

When x_1 is constant, but x_2 varies, the solution is

$$G_{BM} = \begin{cases} \mu_{x_1}^{\alpha} \exp\left\{ \tilde{\mu}_{x_2}\beta + \frac{1}{2}\tilde{\sigma}_{x_2}^2\beta^2 \right\}, \text{ when } \mu_{x_1} \geq 0; \text{ and} \\ 0, \text{ when } \mu_{x_1} < 0. \end{cases} \tag{S42}$$

When x_2 is constant, but x_1 varies, the solution is

$$G_{BM} = \frac{1}{\sqrt{2\pi}} \sigma_{x_1}^{\alpha} \mu_{x_2}^{\beta} \exp\left\{ -\frac{1}{4}\frac{\mu_{x_1}^2}{\sigma_{x_1}^2} \right\} \Gamma(\alpha+1) D_{-(\alpha+1)}\left(-\frac{\mu_{x_1}}{\sigma_{x_1}} \right). \tag{S43}$$

When both x_1 and x_2 are constant ($\sigma_{x_1} = 0$ and $\sigma_{x_2} = 0$), the solution is

$$G_{BM} = \begin{cases} \mu_{x_1}^{\alpha}\mu_{x_2}^{\beta}, \text{ when } \mu_{x_1} \geq 0; \text{ and} \\ 0, \text{ when } \mu_{x_1} < 0. \end{cases} \tag{S44}$$