S1 Modified Analytically Upscaled Microphysics Equations

The analytically upscaled (to grid-box size) form of the Khairoutdinov and Kogan (2000, hereafter KK) microphysics equations were first derived in Larson and Griffin (2013). KK is warm scheme that predicts r_r and N_r. It contains equations for the warm process rates of accretion, autoconversion, and evaporation, as well rain drop mean volume radius, that are written as power laws of two-or-three variables. The modifications to the PDF in Section 2 of the associated article require modifications to the analytically upscaled microphysics equations. The upscaled microphysics calculates the grid-box mean values of microphysics process rates by integrating over the product of the microphysics function and the PDF.

S1.1 Accretion rate

The KK rate of production of r_r over time, t, due to the process of accretion is of the form

$$\left.\frac{\partial r_r}{\partial t}\right|_{\text{accr}} = C_{\text{accr}}\, r_c^{\alpha}\, r_r^{\beta}, \tag{S1}$$

where $C_{\text{accr}} = 67$, $\alpha = 1.15$, and $\beta = 1.15$. Upscaling is accomplished by integrating over Eq. (S1), and in the process, using $r_c = \chi H(\chi)$ (Eq. (3) of the associated article) as a substitution. This produces the following equation for mean accretion rate

$$\begin{aligned} \overline{\left.\frac{\partial r_r}{\partial t}\right|_{\text{accr}}} &= C_{\text{accr}} \sum_{i=1}^{n} \xi_{(i)} \text{ACCR}_{(i)} \\ &= C_{\text{accr}} \sum_{i=1}^{n} \xi_{(i)} \int_{-\infty}^{\infty}\int_{0}^{\infty} \chi^{\alpha} \left(H(\chi)\right)^{\alpha} r_r^{\beta} P_{(i)}(\chi, r_r)\, \mathrm{d}r_r\, \mathrm{d}\chi, \end{aligned} \tag{S2}$$

where $P_{(i)}(\chi, r_r)$ is the bivariate marginal PDF of χ and r_r in the ith PDF component.

Since $\alpha > 0$, $\text{ACCR}_{(i)}$ can be rewritten

$$\text{ACCR}_{(i)} = \int_{0}^{\infty}\int_{0}^{\infty} \chi^{\alpha} r_r^{\beta} \left(f_{p(i)} P_{NL(i)}(\chi, r_r) + \left(1 - f_{p(i)}\right) P_{N(i)}(\chi)\ \delta(r_r) \right) \mathrm{d}r_r\, \mathrm{d}\chi, \tag{S3}$$

where $P_{NL(i)}(\chi, r_r)$ is the ith component bivariate PDF involving one normal variate and one lognormal variate, and where $P_{N(i)}(\chi)$ is a normal distribution in the ith PDF component. This equation is integrated, solving for $\text{ACCR}_{(i)}$,

$$\text{ACCR}_{(i)} = f_{p(i)} \frac{1}{\sqrt{2\pi}}\, \sigma_{\chi(i)}^{\alpha} \exp\left\{ \tilde{\mu}_{r_r(i)}\beta + \frac{1}{2}\tilde{\sigma}_{r_r(i)}^{2}\beta^{2} - \frac{1}{4}\varsigma^{2} \right\} \Gamma(\alpha+1)\, D_{-(\alpha+1)}(-\varsigma), \tag{S4}$$

where $D_{\nu}(x)$ is the parabolic cylinder function of order ν, and where ς is given by

$$\varsigma = \frac{\mu_{\chi(i)}}{\sigma_{\chi(i)}} + \tilde{\rho}_{\chi, r_r(i)} \tilde{\sigma}_{r_r(i)} \beta.$$

The within-precipitation mean of $\ln r_r$ in the ith PDF component is $\tilde{\mu}_{r_r(i)}$, and it is given by

$$\tilde{\mu}_{r_r(i)} = \ln\left(\mu_{r_r(i)}\left(1+\frac{\sigma^2_{r_r(i)}}{\mu^2_{r_r(i)}}\right)^{-\frac{1}{2}}\right), \tag{S5}$$

where $\mu_{r_r(i)}$ and $\sigma_{r_r(i)}$ are the within-precipitation mean and within-precipitation standard deviation, respectively, of r_r in the ith PDF component. The within-precipitation standard deviation of $\ln r_r$ in the ith PDF component is $\tilde{\sigma}_{r_r(i)}$, and it is given by

$$\tilde{\sigma}_{r_r(i)} = \sqrt{\ln\left(1+\frac{\sigma^2_{r_r(i)}}{\mu^2_{r_r(i)}}\right)}. \tag{S6}$$

The within-precipitation correlation of χ and $\ln r_r$ in the ith PDF component is $\tilde{\rho}_{\chi,r_r(i)}$, and it is given by

$$\tilde{\rho}_{\chi,r_r(i)} = \frac{\rho_{\chi,r_r(i)}}{\tilde{\sigma}_{r_r(i)}}\frac{\sigma_{r_r(i)}}{\mu_{r_r(i)}}, \tag{S7}$$

where $\rho_{\chi,r_r(i)}$ is the within-precipitation correlation of χ and r_r in the ith PDF component.

The evaluated integral for $\text{ACCR}_{(i)}$ given in Eq. (S4) is for a fully-varying PDF in the ith component ($\sigma_{\chi(i)} > 0$ and $\sigma_{r_r(i)} > 0$). There are times when a variable may have a constant value in a PDF sub-component. When this happens, the PDF of the constant variable is a delta function at the ith PDF sub-component mean. When $\sigma_{\chi(i)} > 0$ and $\sigma_{r_r(i)} = 0$, χ varies in ith component but r_r is constant within precipitation. The PDF $P_{NL(i)}(\chi, r_r)$ becomes $P_{N(i)}(\chi)\,\delta\left(r_r - \mu_{r_r(i)}\right)$. The integral is solved and the equation for $\text{ACCR}_{(i)}$ becomes

$$\text{ACCR}_{(i)} = f_{p(i)}\frac{1}{\sqrt{2\pi}}\,\sigma^{\alpha}_{\chi(i)}\mu^{\beta}_{r_r(i)}\exp\left\{-\frac{1}{4}\frac{\mu^2_{\chi(i)}}{\sigma^2_{\chi(i)}}\right\}\Gamma(\alpha+1)\,D_{-(\alpha+1)}\left(-\frac{\mu_{\chi(i)}}{\sigma_{\chi(i)}}\right). \tag{S8}$$

For the remaining forms of $\text{ACCR}_{(i)}$, $\sigma_{\chi(i)} = 0$. When $\mu_{\chi(i)} \geq 0$, the air is entirely saturated and accretion occurs. In this scenario, when $\sigma_{r_r(i)} > 0$,

$$\text{ACCR}_{(i)} = f_{p(i)}\mu^{\alpha}_{\chi(i)}\exp\left\{\tilde{\mu}_{r_r(i)}\beta + \frac{1}{2}\tilde{\sigma}^2_{r_r(i)}\beta^2\right\}, \tag{S9}$$

and when $\sigma_{r_r(i)} = 0$,

$$\text{ACCR}_{(i)} = f_{p(i)}\mu^{\alpha}_{\chi(i)}\mu^{\beta}_{r_r(i)}. \tag{S10}$$

Otherwise, when $\sigma_{\chi(i)} = 0$ and $\mu_{\chi(i)} < 0$, the air is entirely subsaturated, accretion does not occur, and $\text{ACCR}_{(i)} = 0$.

S1.2 Autoconversion rate

The KK autoconversion rate of r_r is of the form

$$\left.\frac{\partial r_r}{\partial t}\right|_{\text{auto}} = C_{\text{auto}}\, r_c^{\alpha}\, N_c^{\beta}, \tag{S11}$$

where constant $C_{\text{auto}} = 1350\left(10^{-6}\rho_d\right)^{\beta}$, and where ρ_d is the density of dry air. Additionally, $\alpha = 2.47$ and $\beta = -1.79$. In the manner similar to accretion rate, upscaling is accomplished by integrating over Eq. (S11), and in the process, using $r_c = \chi H(\chi)$ and $N_c = N_{cn} H(\chi)$ (Eq. (3) and Eq. (4) of the associated article) as substitutions. This produces the following equation for mean autoconversion rate

$$\begin{aligned}\overline{\left.\frac{\partial r_r}{\partial t}\right|_{\text{auto}}} &= C_{\text{auto}} \sum_{i=1}^{n} \xi_{(i)} \text{AUTO}_{(i)} \\ &= C_{\text{auto}} \sum_{i=1}^{n} \xi_{(i)} \int_{-\infty}^{\infty} \int_{0}^{\infty} \chi^{\alpha} N_{cn}^{\beta} \left(H(\chi)\right)^{\alpha+\beta} P_{(i)}(\chi, N_{cn})\, \mathrm{d}N_{cn}\, \mathrm{d}\chi.\end{aligned} \tag{S12}$$

Since $\alpha + \beta > 0$, $\text{AUTO}_{(i)}$ can be rewritten

$$\text{AUTO}_{(i)} = \int_{0}^{\infty} \int_{0}^{\infty} \chi^{\alpha} N_{cn}^{\beta}\, P_{NL(i)}(\chi, N_{cn})\, \mathrm{d}N_{cn}\, \mathrm{d}\chi. \tag{S13}$$

This equation is integrated, solving for $\text{AUTO}_{(i)}$ in the scenario of a fully-varying PDF ($\sigma_{\chi(i)} > 0$ and $\sigma_{N_{cn}(i)} > 0$),

$$\text{AUTO}_{(i)} = \frac{1}{\sqrt{2\pi}}\, \sigma_{\chi(i)}^{\alpha} \exp\left\{\tilde{\mu}_{N_{cn}(i)}\beta + \frac{1}{2}\tilde{\sigma}_{N_{cn}(i)}^{2}\beta^{2} - \frac{1}{4}\varsigma^{2}\right\} \Gamma(\alpha+1)\, D_{-(\alpha+1)}(-\varsigma), \tag{S14}$$

where ς is given by

$$\varsigma = \frac{\mu_{\chi(i)}}{\sigma_{\chi(i)}} + \tilde{\rho}_{\chi,N_{cn}(i)} \tilde{\sigma}_{N_{cn}(i)} \beta.$$

The values of $\tilde{\mu}_{N_{cn}(i)}$, $\tilde{\sigma}_{N_{cn}(i)}$, and $\tilde{\rho}_{\chi,N_{cn}(i)}$ are calculated analogously to the same variables for r_r in Eq. (S5), Eq. (S6), and Eq. (S7), respectively.

There are many case specifications that require a constant cloud droplet concentration within cloud, N_{c0}. The RICO, DYCOMS-II RF02, and LBA cases described in Section 4 of the associated article all use a constant cloud droplet concentration within cloud. In CLUBB's PDF, this is easily accomplished by setting $\overline{N_{cn}'^{2}} = 0$, which causes $\sigma_{N_{cn}(1)} = 0$ and $\sigma_{N_{cn}(2)} = 0$. Additionally, $\mu_{N_{cn}(1)} = \mu_{N_{cn}(2)} = \overline{N_{cn}} = N_{c0}$ (where N_{c0} has units of kg^{-1}) in this scenario.

When $\sigma_{\chi(i)} > 0$ and $\sigma_{N_{cn}(i)} = 0$, χ varies in ith component but N_{cn} is constant. The integral is solved and the equation for $\text{AUTO}_{(i)}$ becomes

$$\text{AUTO}_{(i)} = \frac{1}{\sqrt{2\pi}}\, \sigma_{\chi(i)}^{\alpha} \mu_{N_{cn}(i)}^{\beta} \exp\left\{-\frac{1}{4}\frac{\mu_{\chi(i)}^{2}}{\sigma_{\chi(i)}^{2}}\right\} \Gamma(\alpha+1)\, D_{-(\alpha+1)}\left(-\frac{\mu_{\chi(i)}}{\sigma_{\chi(i)}}\right). \tag{S15}$$

For the remaining forms of $\text{AUTO}_{(i)}$, $\sigma_{\chi(i)} = 0$. When $\mu_{\chi(i)} \geq 0$, the air is entirely saturated and autoconversion occurs. In this scenario, when $\sigma_{N_{cn}(i)} > 0$,

$$\text{AUTO}_{(i)} = \mu_{\chi(i)}^{\alpha} \exp\left\{\tilde{\mu}_{N_{cn}(i)}\beta + \frac{1}{2}\tilde{\sigma}_{N_{cn}(i)}^{2}\beta^{2}\right\}, \tag{S16}$$

and when $\sigma_{N_{cn}(i)} = 0$,

$$\text{AUTO}_{(i)} = \mu_{\chi(i)}^{\alpha} \mu_{N_{cn}(i)}^{\beta}. \tag{S17}$$

Otherwise, when $\sigma_{\chi(i)} = 0$ and $\mu_{\chi(i)} < 0$, the air is entirely subsaturated, autoconversion does not occur, and $\text{AUTO}_{(i)} = 0$.

The mean KK autoconversion rate of N_r is found by dividing the mean KK autoconversion rate of r_r by a constant. The constant is $(4\pi\rho_l/3)\, r_0^3$, where r_0 is the assumed initial size of rain drops and is set to its recommended value of 25×10^{-6} m.

S1.3 Evaporation rate

The KK equation set contains an equation for condensation or evaporation of r_r. CLUBB treats all liquid water in excess of saturation as cloud water and does not allow rain water to increase by condensational growth. The KK equation for evaporation of r_r is of the form

$$\left.\frac{\partial r_r}{\partial t}\right|_{\text{evap}} = 3\, c_{\text{evap}*} G\,(T,p) \left(\frac{4}{3}\pi\rho_l\right)^{\gamma} (S\, H\,(-S))^{\alpha}\, r_r^{\beta} N_r^{\gamma}, \tag{S18}$$

where $\alpha = 1$, $\beta = 1/3$, and $\gamma = 1 - \beta = 2/3$, and where T is temperature, p is pressure, ρ_l is the density of liquid water, and the function $G\,(T,p)$ is coefficient in the drop radius growth equation (Rogers and Yau, 1989, Eq. 7.17). The constant $c_{\text{evap}*}$ is the ratio of raindrop mean geometric radius to raindrop mean volume radius, and is set by KK to a value of 0.86. Supersaturation, S, is positive when the air is saturated and negative when the air is subsaturated, and $S+1$ is the ratio of water vapor pressure to saturation vapor pressure with respect to liquid water.

Upscaling is accomplished by integrating over Eq. (S18). This requires a substitution that relates S to χ (Larson and Griffin, 2013, Eq. 49). Additionally, $G\,(T,p)$ is approximated as $G\left(\overline{T_l},p\right)$, where liquid water temperature, $T_l = \theta_l\,(p/p_0)^{R_d/c_{pd}}$, and where R_d is the gas constant for dry air, c_{pd} is the specific heat of dry air at constant pressure, and p_0 is a reference pressure of 1×10^5 Pa. This is a good approximation because $T = T_l$ when the air is subsaturated and $G\,(T,p)$ is slowly varying with regards to temperature. The resulting $G\left(\overline{T_l},p\right)$ is a constant and can be pulled outside the integral.

This produces the following equation for mean evaporation rate

$$\begin{aligned} \overline{\left.\frac{\partial r_r}{\partial t}\right|_{\text{evap}}} &= C_{\text{evap}} \sum_{i=1}^{n} \xi_{(i)} \text{EVAP}_{(i)} \\ &= C_{\text{evap}} \sum_{i=1}^{n} \xi_{(i)} \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} \chi^{\alpha}\,(H\,(-\chi))^{\alpha}\, r_r^{\beta} N_r^{\gamma} P_{(i)}\,(\chi, r_r, N_r)\, \mathrm{d}N_r\, \mathrm{d}r_r\, \mathrm{d}\chi, \end{aligned} \tag{S19}$$

where $P_{(i)}\,(\chi, r_r, N_r)$ is the trivariate marginal PDF of χ, r_r, N_r in the ith PDF component. The constant C_{evap} is given by

$$C_{\text{evap}} = 3\, c_{\text{evap}*} G\left(\overline{T_l},p\right) \left(\frac{4}{3}\pi\rho_l\right)^{\gamma} \left(\frac{1 + \Lambda\left(\overline{T_l}\right) r_{sw}\left(\overline{T_l},p\right)}{r_{sw}\left(\overline{T_l},p\right)}\right)^{\alpha}, \tag{S20}$$

where $r_{sw}\left(\overline{T_l}, p\right)$ is the saturation mixing ratio with respect to liquid water. Additionally,

$$\Lambda\left(\overline{T_l}\right) = \frac{R_d}{R_v}\left(\frac{L_v}{R_d\overline{T_l}}\right)\left(\frac{L_v}{c_{pd}\overline{T_l}}\right), \tag{S21}$$

where R_v is the gas constant for water vapor and L_v is the latent heat of vaporization.

Since $\alpha > 0$, $\mathrm{EVAP}_{(i)}$ can be rewritten

$$\begin{aligned}\mathrm{EVAP}_{(i)} = \int_{-\infty}^{0}\int_{0}^{\infty}\int_{0}^{\infty} \chi^{\alpha} r_r^{\beta} N_r^{\gamma} \Big(& f_{p(i)} P_{NLL(i)}\left(\chi, r_r, N_r\right) \\ & + \left(1 - f_{p(i)}\right) P_{N(i)}\left(\chi\right) \delta\left(r_r\right) \delta\left(N_r\right)\Big) \,\mathrm{d}N_r \,\mathrm{d}r_r \,\mathrm{d}\chi,\end{aligned} \tag{S22}$$

where $P_{NLL(i)}\left(\chi, r_r, N_r\right)$ is the ith component trivariate PDF involving one normal variate and two lognormal variates. When the PDF is fully-varying in the ith PDF component ($\sigma_{\chi(i)} > 0$, $\sigma_{r_r(i)} > 0$, and $\sigma_{N_r(i)} > 0$), the integrated equation for $\mathrm{EVAP}_{(i)}$ is

$$\begin{aligned}\mathrm{EVAP}_{(i)} = {} & f_{p(i)} \frac{1}{\sqrt{2\pi}} \left(-\sigma_{\chi(i)}\right)^{\alpha} \exp\left\{\tilde{\mu}_{r_r(i)}\beta + \tilde{\mu}_{N_r(i)}\gamma\right\} \\ & \times \exp\Big\{\frac{1}{2}\left(1 - \tilde{\rho}_{\chi,r_r(i)}^2\right)\tilde{\sigma}_{r_r(i)}^2\beta^2 + \frac{1}{2}\left(1 - \tilde{\rho}_{\chi,N_r(i)}^2\right)\tilde{\sigma}_{N_r(i)}^2\gamma^2 \\ & \qquad + \left(\tilde{\rho}_{r_r,N_r(i)} - \tilde{\rho}_{\chi,r_r(i)}\tilde{\rho}_{\chi,N_r(i)}\right)\tilde{\sigma}_{r_r(i)}\beta\tilde{\sigma}_{N_r(i)}\gamma\Big\} \\ & \times \exp\left\{\frac{1}{4}\varsigma^2 - \frac{\mu_{\chi(i)}}{\sigma_{\chi(i)}}\varsigma + \frac{1}{2}\frac{\mu_{\chi(i)}^2}{\sigma_{\chi(i)}^2}\right\}\Gamma\left(\alpha+1\right) D_{-(\alpha+1)}\left(\varsigma\right),\end{aligned} \tag{S23}$$

where

$$\varsigma = \frac{\mu_{\chi(i)}}{\sigma_{\chi(i)}} + \tilde{\rho}_{\chi,r_r(i)}\tilde{\sigma}_{r_r(i)}\beta + \tilde{\rho}_{\chi,N_r(i)}\tilde{\sigma}_{N_r(i)}\gamma.$$

The values of $\tilde{\mu}_{N_r(i)}$, $\tilde{\sigma}_{N_r(i)}$, and $\tilde{\rho}_{\chi,N_r(i)}$ are calculated analogously to the same variables for r_r in Eq. (S5), Eq. (S6), and Eq. (S7), respectively. Additionally, the within-precipitation correlation of $\ln r_r$ and $\ln N_r$ in the ith PDF component is $\tilde{\rho}_{r_r,N_r(i)}$, and it is given by

$$\tilde{\rho}_{r_r,N_r(i)} = \frac{\ln\left(1 + \rho_{r_r,N_r(i)}\dfrac{\sigma_{r_r(i)}\sigma_{N_r(i)}}{\mu_{r_r(i)}\mu_{N_r(i)}}\right)}{\tilde{\sigma}_{r_r(i)}\tilde{\sigma}_{N_r(i)}}, \tag{S24}$$

where $\rho_{r_r,N_r(i)}$ is the correlation of r_r and N_r in the ith PDF component.

Just as with accretion and autoconversion, when one of the variables is constant in the ith PDF sub-component, the equation simplifies. In the scenario when $\sigma_{\chi(i)} > 0$, $\sigma_{r_r(i)} = 0$, and $\sigma_{N_r(i)} > 0$,

$$\begin{aligned}\mathrm{EVAP}_{(i)} = {} & f_{p(i)} \frac{1}{\sqrt{2\pi}} \left(-\sigma_{\chi(i)}\right)^{\alpha} \mu_{r_r(i)}^{\beta} \\ & \times \exp\left\{\tilde{\mu}_{N_r(i)}\gamma + \frac{1}{2}\tilde{\sigma}_{N_r(i)}^2\gamma^2 - \frac{1}{4}\varsigma^2\right\}\Gamma\left(\alpha+1\right) D_{-(\alpha+1)}\left(\varsigma\right),\end{aligned} \tag{S25}$$

where

$$\varsigma = \frac{\mu_{\chi(i)}}{\sigma_{\chi(i)}} + \tilde{\rho}_{\chi,N_r(i)}\tilde{\sigma}_{N_r(i)}\gamma;$$

when $\sigma_{\chi(i)} > 0$, $\sigma_{r_r(i)} > 0$, and $\sigma_{N_r(i)} = 0$,

$$\begin{aligned} \mathrm{EVAP}_{(i)} = {} & f_{p(i)} \frac{1}{\sqrt{2\pi}} \left(-\sigma_{\chi(i)}\right)^{\alpha} \mu_{N_r(i)}^{\gamma} \\ & \times \exp\left\{ \tilde{\mu}_{r_r(i)}\beta + \frac{1}{2}\tilde{\sigma}_{r_r(i)}^2\beta^2 - \frac{1}{4}\varsigma^2 \right\} \Gamma\left(\alpha+1\right) D_{-(\alpha+1)}\left(\varsigma\right), \end{aligned} \tag{S26}$$

where

$$\varsigma = \frac{\mu_{\chi(i)}}{\sigma_{\chi(i)}} + \tilde{\rho}_{\chi,r_r(i)}\tilde{\sigma}_{r_r(i)}\beta;$$

and when $\sigma_{\chi(i)} > 0$, $\sigma_{r_r(i)} = 0$, and $\sigma_{N_r(i)} = 0$,

$$\begin{aligned} \mathrm{EVAP}_{(i)} = {} & f_{p(i)} \frac{1}{\sqrt{2\pi}} \left(-\sigma_{\chi(i)}\right)^{\alpha} \mu_{r_r(i)}^{\beta}\mu_{N_r(i)}^{\gamma} \\ & \times \exp\left\{ -\frac{1}{4}\frac{\mu_{\chi(i)}^2}{\sigma_{\chi(i)}^2} \right\} \Gamma\left(\alpha+1\right) D_{-(\alpha+1)}\left(\frac{\mu_{\chi(i)}}{\sigma_{\chi(i)}}\right). \end{aligned} \tag{S27}$$

For the remaining forms of $\mathrm{EVAP}_{(i)}$, $\sigma_{\chi(i)} = 0$. When $\mu_{\chi(i)} \leq 0$, the air is entirely subsaturated and evaporation occurs. In this scenario, when $\sigma_{r_r(i)} > 0$ and $\sigma_{N_r(i)} > 0$,

$$\begin{aligned} \mathrm{EVAP}_{(i)} = f_{p(i)}\mu_{\chi(i)}^{\alpha} \exp\Big\{ & \tilde{\mu}_{r_r(i)}\beta + \tilde{\mu}_{N_r(i)}\gamma + \frac{1}{2}\tilde{\sigma}_{r_r(i)}^2\beta^2 \\ & + \frac{1}{2}\tilde{\sigma}_{N_r(i)}^2\gamma^2 + \tilde{\rho}_{r_r,N_r(i)}\tilde{\sigma}_{r_r(i)}\beta\tilde{\sigma}_{N_r(i)}\gamma \Big\}; \end{aligned} \tag{S28}$$

when $\sigma_{r_r(i)} = 0$ and $\sigma_{N_r(i)} > 0$,

$$\mathrm{EVAP}_{(i)} = f_{p(i)}\mu_{\chi(i)}^{\alpha}\mu_{r_r(i)}^{\beta} \exp\left\{ \tilde{\mu}_{N_r(i)}\gamma + \frac{1}{2}\tilde{\sigma}_{N_r(i)}^2\gamma^2 \right\}; \tag{S29}$$

when $\sigma_{r_r(i)} > 0$ and $\sigma_{N_r(i)} = 0$,

$$\mathrm{EVAP}_{(i)} = f_{p(i)}\mu_{\chi(i)}^{\alpha}\mu_{N_r(i)}^{\gamma} \exp\left\{ \tilde{\mu}_{r_r(i)}\beta + \frac{1}{2}\tilde{\sigma}_{r_r(i)}^2\beta^2 \right\}; \tag{S30}$$

and when $\sigma_{r_r(i)} = 0$ and $\sigma_{N_r(i)} = 0$,

$$\mathrm{EVAP}_{(i)} = f_{p(i)}\mu_{\chi(i)}^{\alpha}\mu_{r_r(i)}^{\beta}\mu_{N_r(i)}^{\gamma}. \tag{S31}$$

Otherwise, when $\sigma_{\chi(i)} = 0$ and $\mu_{\chi(i)} > 0$, the air is entirely saturated, evaporation does not occur, and $\mathrm{EVAP}_{(i)} = 0$.

The KK evaporation rate of N_r is related to the evaporation rate of r_r by

$$\frac{\Delta N_r|_{\mathrm{evap}}}{N_r} = \left(\frac{\Delta r_r|_{\mathrm{evap}}}{r_r}\right)^{\nu_*}, \tag{S32}$$

where $\Delta N_r|_{\text{evap}}$ is the change in N_r due to evaporation, $\Delta r_r|_{\text{evap}}$ is the change in r_r due to evaporation, and ν_* is a tunable parameter in KK that is set to its recommended value of 1. CLUBB does not handle microphysics process rates in a sequential manner, but rather in a parallel manner. However, the microphysics process rates are explicit terms in the predictive equation set, so the change in a hydrometeor due to a microphysics process is related to the rate of change by

$$\Delta r_r|_{\text{evap}} = \left.\frac{\partial r_r}{\partial t}\right|_{\text{evap}} \Delta t \qquad \text{and} \qquad \Delta N_r|_{\text{evap}} = \left.\frac{\partial N_r}{\partial t}\right|_{\text{evap}} \Delta t, \tag{S33}$$

where Δt is the duration of one model timestep. Substituting Eq. (S33) into Eq. (S32) and solving for the rate of change of N_r due to evaporation results in

$$\left.\frac{\partial N_r}{\partial t}\right|_{\text{evap}} = (\Delta t)^{\nu_*-1} \frac{N_r}{{r_r}^{\nu_*}} \left(\left.\frac{\partial r_r}{\partial t}\right|_{\text{evap}} \right)^{\nu_*}. \tag{S34}$$

The mean N_r evaporation rate is calculated in the same way as the mean r_r evaporation rate with α replaced by $\alpha\nu_*$, β replaced by $(\beta - 1)\,\nu_*$, and γ replaced by $\gamma\nu_* + 1$. Additionally, the constant C_{evap} is taken to the ν_* power and the result is multiplied by $(\Delta t)^{\nu_*-1}$. When ν_* is set to its recommended value of 1, the mean N_r evaporation rate is more simply solved the same way as the mean r_r evaporation rate with β replaced by $\beta - 1$ and γ replaced by $\gamma + 1$.

S1.4 Mean volume radius of rain drops

The KK mean volume radius of rain drops (in meters), R_{vr}, is of the form

$$R_{vr} = C_{\text{mvrr}}\, r_r^{\alpha}\, N_r^{\beta}, \tag{S35}$$

where $C_{\text{mvrr}} = (4\pi\rho_l/3)^{\beta}$, $\alpha = 1/3$, and $\beta = -\alpha = -1/3$. Upscaling is accomplished by integrating over Eq. (S35), producing the following equation for mean volume radius

$$\begin{aligned} \overline{R_{vr}} &= C_{\text{mvrr}} \sum_{i=1}^{n} \xi_{(i)} \text{MVRR}_{(i)} \\ &= C_{\text{mvrr}} \sum_{i=1}^{n} \xi_{(i)} \int_0^{\infty}\!\!\int_0^{\infty} r_r^{\alpha} N_r^{\beta} P_{(i)}\left(r_r, N_r\right) \mathrm{d}N_r\, \mathrm{d}r_r, \end{aligned} \tag{S36}$$

where $P_{(i)}\left(r_r, N_r\right)$ is the bivariate marginal PDF of r_r and N_r in the ith PDF component.

Additionally, $\text{MVRR}_{(i)}$ can be rewritten

$$\text{MVRR}_{(i)} = \int_0^{\infty}\!\!\int_0^{\infty} r_r^{\alpha} N_r^{\beta} \left(f_{p(i)} P_{LL(i)}\left(r_r, N_r\right) + \left(1 - f_{p(i)}\right) \delta\left(r_r\right) \delta\left(N_r\right) \right) \mathrm{d}N_r\, \mathrm{d}r_r, \tag{S37}$$

where $P_{LL(i)}\left(r_r, N_r\right)$ is the ith component bivariate PDF involving two lognormal variates. When the PDF is fully-varying in the ith PDF component ($\sigma_{r_r(i)} > 0$ and $\sigma_{N_r(i)} > 0$), the integrated equation for $\mathrm{MVRR}_{(i)}$ is

$$\begin{aligned}\mathrm{MVRR}_{(i)} =& f_{p(i)} \exp\left\{ \tilde{\mu}_{r_r(i)}\alpha + \tilde{\mu}_{N_r(i)}\beta + \frac{1}{2}\tilde{\sigma}^2_{r_r(i)}\alpha^2 \right. \\ & \left. + \frac{1}{2}\tilde{\sigma}^2_{N_r(i)}\beta^2 + \tilde{\rho}_{r_r,N_r(i)}\tilde{\sigma}_{r_r(i)}\alpha\tilde{\sigma}_{N_r(i)}\beta \right\}.\end{aligned} \tag{S38}$$

In the scenario when $\sigma_{r_r(i)} = 0$ and $\sigma_{N_r(i)} > 0$,

$$\mathrm{MVRR}_{(i)} = f_{p(i)}\mu^{\alpha}_{r_r(i)} \exp\left\{ \tilde{\mu}_{N_r(i)}\beta + \frac{1}{2}\tilde{\sigma}^2_{N_r(i)}\beta^2 \right\}, \tag{S39}$$

when $\sigma_{r_r(i)} > 0$ and $\sigma_{N_r(i)} = 0$,

$$\mathrm{MVRR}_{(i)} = f_{p(i)}\mu^{\beta}_{N_r(i)} \exp\left\{ \tilde{\mu}_{r_r(i)}\alpha + \frac{1}{2}\tilde{\sigma}^2_{r_r(i)}\alpha^2 \right\}, \tag{S40}$$

and when $\sigma_{r_r(i)} = 0$ and $\sigma_{N_r(i)} = 0$,

$$\mathrm{MVRR}_{(i)} = f_{p(i)}\mu^{\alpha}_{r_r(i)}\mu^{\beta}_{N_r(i)}. \tag{S41}$$

The upscaled mean volume radius is used to calculate mean sedimentation velocity of r_r and N_r. The mean sedimentation velocity of r_r is $\overline{V_{r_r}} = \min\left(-0.012\left(10^6\,\overline{R_{vr}}\right) + 0.2, 0\right)$, and the mean sedimentation velocity of N_r is $\overline{V_{N_r}} = \min\left(-0.007\left(10^6\,\overline{R_{vr}}\right) + 0.1, 0\right)$.

References

Khairoutdinov, M. and Kogan, Y.: A new cloud physics parameterization in a large-eddy simulation model of marine stratocumulus, Mon. Wea. Rev., 128, 229–243, 2000.

Larson, V. E. and Griffin, B. M.: Analytic upscaling of local microphysics parameterizations, Part I: Derivation, Quart. J. Royal Met. Soc., 139, 46–57, 2013.

Rogers, R. R. and Yau, M. K.: A short course in cloud physics, Butterworth-Heinemann, 3rd edn., 1989.