Mathematical details to find $\mathbf{Q}$ for a 2×2 lattice.

Consider $\mathbf{x} \sim N(\mathbf{0}_{4\times 1}, \mathbf{Q}_{4\times 4})$, a vector of measurements on a 2×2 lattice, as represented in Figure 1 of the main manuscript. Assume a neighborhood structure between the four elements of $\mathbf{x}$. In Figure 2 of the main manuscript, the neighbors for each element of $\mathbf{x}$ are defined graphically. Given this structure, one can write expressions for the conditional means that reflect how information at each grid point might be related to its neighbors. Therefore,

$$E(x_1|x_2, x_3, x_4) = \beta_{12}x_2 + \beta_{13}x_3, \tag{1}$$

$$E(x_2|x_1, x_3, x_4) = \beta_{21}x_1 + \beta_{24}x_4, \tag{2}$$

$$E(x_3|x_1, x_2, x_4) = \beta_{31}x_1 + \beta_{34}x_4, \tag{3}$$

$$E(x_4|x_1, x_2, x_3) = \beta_{42}x_2 + \beta_{43}x_3. \tag{4}$$

These expressions are used to find a relationship between the β coefficients and the elements of $\mathbf{Q}$. Since $\mathbf{x} \sim N(\mathbf{0}_{4\times 1}, \mathbf{Q}_{4\times 4})$, the joint probability distribution of $\mathbf{x}$ is given by,

$$\begin{aligned} f(x_1, x_2, x_3, x_4) \propto \quad & exp(-\tfrac{1}{2}(Q_{11}x_1^2 + Q_{22}x_2^2 + Q_{33}x_3^2 + Q_{44}x_4^2 + 2Q_{12}x_1x_2 + 2Q_{13}x_1x_3 + 2Q_{14}x_1x_4 \\ & +2Q_{23}x_2x_3 + 2Q_{24}x_2x_4 + 2Q_{34}x_3x_4)). \end{aligned}$$

Using this joint probability distribution, we derive the full conditional of x_1 given x_2, x_3, x_4,

$$f(x_1|x_2, x_3, x_4) \propto exp\left\{-\frac{1}{2}Q_{11}\left(x_1^2 - 2x_1\left(-\frac{Q_{12}}{Q_{11}}x_2 - \frac{Q_{13}}{Q_{11}}x_3 - \frac{Q_{14}}{Q_{11}}x_4\right)\right)\right\}. \tag{5}$$

This expression can be re-written as

$$f(x_1|x_2, x_3, x_4) \propto exp\left\{-\frac{1}{2}Q_{11}\left(x_1 - \left(-\frac{Q_{12}}{Q_{11}}x_2 - \frac{Q_{13}}{Q_{11}}x_3 - \frac{Q_{14}}{Q_{11}}x_4\right)\right)^2\right\}. \tag{6}$$

From matching (6) to the expression of a univariate normal distribution,

$$E(x_1|x_2, x_3, x_4) = -\frac{Q_{12}}{Q_{11}}x_2 - \frac{Q_{13}}{Q_{11}}x_3 - \frac{Q_{14}}{Q_{11}}x_4, \tag{7}$$

and

$$Prec(x_1|x_2, x_3, x_4) = Q_{11}. \tag{8}$$

By comparing equations (1) and (7), we obtain

$$\beta_{12} = -\frac{Q_{12}}{Q_{11}}, \qquad \beta_{13} = -\frac{Q_{13}}{Q_{11}}, \qquad \beta_{14} = -\frac{Q_{14}}{Q_{11}} = 0.$$

Considering the full conditionals for x_2, x_3 and x_4 and its conditional expectations respectively, yield similar relationships between the β coefficients and the elements of $\mathbf{Q}$:

$$\beta_{21} = -\frac{Q_{21}}{Q_{22}}, \qquad \beta_{23} = -\frac{Q_{23}}{Q_{22}} = 0, \qquad \beta_{24} = -\frac{Q_{24}}{Q_{44}}$$

$$\beta_{31} = -\frac{Q_{31}}{Q_{33}}, \qquad \beta_{32} = -\frac{Q_{32}}{Q_{33}} = 0, \qquad \beta_{34} = -\frac{Q_{34}}{Q_{33}}$$

$$\beta_{41} = -\frac{Q_{41}}{Q_{44}} = 0, \qquad \beta_{42} = -\frac{Q_{42}}{Q_{44}}, \qquad \beta_{43} = -\frac{Q_{43}}{Q_{44}}.$$

These relationships hold for an n dimensional distribution as established in Rue and Held [1]. If the conditional means and precisions can be written as

$$E(x_i|x_{-i}) = \mu_i + \sum_{j \neq i} \beta_{ij}(x_j - \mu_j) \quad \text{and} \tag{9}$$

$$Prec(x_i|x_{-i}) = k_i > 0, \tag{10}$$

then $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ follows a multivariate normal distribution with mean $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$ and precision matrix $\mathbf{Q}$ of entries Q_{ij}, where

$$Q_{ij} = \begin{cases} -k_i\beta_{ij} & i \neq j \\ k_i & i = j \end{cases} \tag{11}$$

provided $k_i\beta_{ij} = k_j\beta_{ji}$, $i \neq j$.

If we let $Prec(x_i|x_{-i}) = 2$ $(i = 1, 2, 3, 4)$, $\beta_{12} = \beta_{13} = \beta_{21} = \beta_{24} = \beta_{31} = \beta_{34} = \beta_{42} = \beta_{43} = 1/2$ and $\beta_{14} = \beta_{23} = \beta_{32} = \beta_{41} = 0$ using equations (9)-(11), $\mathbf{x} = (x_1, x_2, x_3, x_4)$ follows a multivariate normal distribution with mean $\mu = (0, 0, 0, 0)^T$ and precision matrix

$$\mathbf{Q} = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}.$$

Generalizing Q to deal with multiple fields.

The generalization of $\mathbf{Q}$ to handle multiple fields is illustrated by a case with two fields, $\mathbf{x}$ and $\mathbf{y}$ which represent the difference between a model and observations for these fields. These observations are assumed to be on a 2×2 lattice, as shown in Figure 1.

Field One

X_1	X_2
X_3	X_4

Field Two

Y_1	Y_2
Y_3	Y_4

Figure 1: Two fields with observations $\mathbf{x}$, $\mathbf{y}$ defined on a 2×2 lattice.

Firstly $\mathbf{x}$ and $\mathbf{y}$ are combined into one vector $\mathbf{v}$ so that $\mathbf{v} = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8)^T = (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4)^T$. The covariance among these observations can be represented by a 2×2 matrix between the field 1, $\mathbf{x}$, and the field 2, $\mathbf{y}$,

$$\mathbf{S} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix},$$

where $Var(\mathbf{x}) = \sigma_{11}$, $Var(\mathbf{y}) = \sigma_{22}$, and $Cov(\mathbf{x}, \mathbf{y}) = \sigma_{12}$. Recalling that the correlation between fields 1 and 2 is defined as: $\rho = \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}$, it is easy to verify that the inverse of $\mathbf{S}$ is

$$\mathbf{S}^{-1} = \begin{pmatrix} \frac{1}{\sigma_{11}(1-\rho^2)} & \frac{-\rho}{(1-\rho^2)\sqrt{\sigma_{11}\sigma_{22}}} \\ \frac{-\rho}{(1-\rho^2)\sqrt{\sigma_{11}\sigma_{22}}} & \frac{1}{\sigma_{11}(1-\rho^2)} \end{pmatrix}.$$

Defining $\mathbf{Q}^*$ as $\mathbf{S}^{-1} \otimes \mathbf{Q}$, the Kronecker product of $\mathbf{S}^{-1}$ and $\mathbf{Q}$, then,

$$\mathbf{Q}^* = \mathbf{S}^{-1} \otimes \mathbf{Q} = \begin{pmatrix} \frac{1}{\sigma_{11}(1-\rho^2)}\mathbf{Q} & \frac{-\rho}{(1-\rho^2)\sqrt{\sigma_{11}\sigma_{22}}}\mathbf{Q} \\ \frac{-\rho}{(1-\rho^2)\sqrt{\sigma_{11}\sigma_{22}}}\mathbf{Q} & \frac{1}{\sigma_{11}(1-\rho^2)}\mathbf{Q} \end{pmatrix}.$$

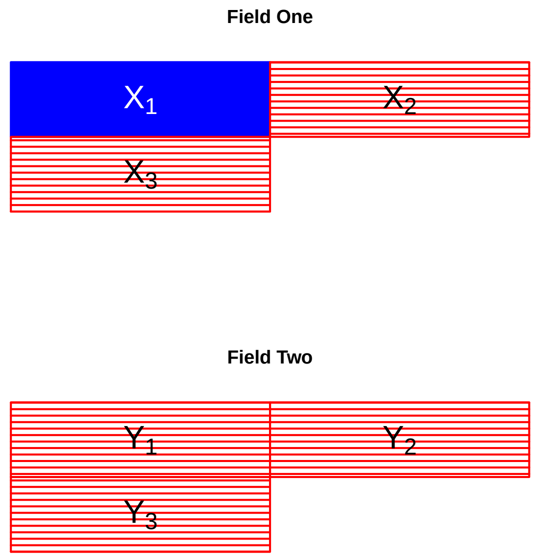

Figure 2: Neighbors of x_1 for a 2×2 lattice and two fields $\mathbf{x}$ and $\mathbf{y}$.

To see what type of relationships are imposed by $\mathbf{Q}^*$ on the elements of $\mathbf{v}$, consider the first element $v_1 = x_1$. Also notice that the first row of $\mathbf{Q}^*$ is,

$$\left(\begin{array}{cccccccc} \frac{2}{\sigma_{11}(1-\rho^2)} & \frac{-1}{(1-\rho^2)\sigma_{11}} & \frac{-1}{(1-\rho^2)\sigma_{11}} & 0 & \frac{-2\rho}{\sqrt{\sigma_{11}\sigma_{22}}(1-\rho^2)} & \frac{\rho}{\sqrt{\sigma_{11}\sigma_{22}}(1-\rho^2)} & \frac{\rho}{\sqrt{\sigma_{11}\sigma_{22}}(1-\rho^2)} & 0 \end{array} \right). \tag{12}$$

Using equations (9)-(11), it can be easily checked that the value for $\beta_{12} = \frac{-Q^*_{12}}{Q^*_{11}} = \frac{1}{2}$. The other β values can be determined in a similar fashion. Using these β coefficients, the equations for the conditional mean and precision of $v_1 = x_1$ given the rest of the elements of $\mathbf{v}$ are

$$E(v_1|\mathbf{v_{-1}}) = \frac{1}{2}x_2 + \frac{1}{2}x_3 + \frac{\rho\sigma_{11}}{\sqrt{\sigma_{11}\sigma_{22}}}y_1 - \frac{\rho\sigma_{11}}{2\sqrt{\sigma_{11}\sigma_{22}}}y_2 - \frac{\rho\sigma_{11}}{2\sqrt{\sigma_{11}\sigma_{22}}}y_3 \tag{13}$$

and

$$Prec(v_1|\mathbf{v_{-1}}) = \frac{1}{\sigma_{11}(1-\rho^2)}. \tag{14}$$

The expression for the conditional mean can be rewritten in terms of the slope b of the linear regression between $\mathbf{x}$ and $\mathbf{y}$, $b = \frac{\rho\sqrt{\sigma_{11}}}{\sqrt{\sigma_{22}}}$, with ρ equal to their correlation,

$$E(v_1|\mathbf{v_{-1}}) = \frac{1}{2}x_2 + \frac{1}{2}x_3 + by_1 - \frac{b}{2}y_2 - \frac{b}{2}y_3. \tag{15}$$

Equation (15) implies that the neighbors of x_1 are $x_2,\ x_3,\ y_1,\ y_2$ and y_3. Figure 2 shows a graphical display of all neighbors of x_1 in the context of the two fields $\mathbf{x}$, $\mathbf{y}$ and a 2×2 lattice.

References Cited

[1] Rue, H. and Held, L.: Gaussian Markov random fields, vol. 104, Chapman & Hall/CRC, 2005.