

## ***Interactive comment on “A global finite-element shallow-water model supporting continuous and discontinuous elements” by P. A. Ullrich***

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*p1543 “This reduced communication requirement implies better overall scalability on large-scale parallel systems.” I worry about remarks like this because once they appear in the literature they tend to get cited out of context. Since a scaling factor is applied to the number of communications, the scaling itself won’t be affected, but the limit to strong scaling might be. However, DG will hit the wall in strong scaling almost as soon as CG does.*

Agreed. Statements such as these have been circulated in the literature for quite some time, but I’m not sure there is any observable effect until one gets to the scale of one element per processor. This sentence will be removed.

C1583

*p1543 “I don’t think that Thuburn (2008) actually says anywhere that the nonconservative form is better for conserving potential enstrophy and angular momentum. Potential enstrophy can be exactly conserved if one specifically uses the vector invariant form of the equations together with a careful treatment of the discretisation, but I don’t think this would necessarily be a generic property of non-conservative formulations. Angular momentum can only be exactly preserved if the grid has rotational symmetries.*

The phrasing of this sentence is understandably confusing: I mean to say that Thuburn (2008) describes angular momentum and potential enstrophy as quantities which are relevant for atmospheric motions. This form of the equations is meant to be contrasted with the conservative form, which evolves the momentum  $hu^d$ . In conservative form / flux-form it is unclear how these secondary quantities can be conserved (without choosing them as prognostic variables explicitly). This sentence has been rephrased as “Angular momentum and potential enstrophy are particularly relevant to atmospheric motions (Thuburn, 2008). When these variables are treated as diagnostic quantities, it is unclear how they can be conserved when the fluid equations are formulated in flux-form, as is typical for the discontinuous Galerkin formulation (Nair, 2005). The non-conservative formulation also has the advantage of leading to a more accurate treatment of wave-like motion when formulated appropriately (Thuburn and Woollings, 2005).

*p5148: please provide more clues as to why this discretisation produces identical results to nodal spectral element for CG spaces i.e. with direct stiffness summation.*

For continuous elements,  $\bar{f} = f$  and (15) reduces to

$$D_{\alpha}f(\alpha_i, \beta_j) = \sum_{p=0}^{n_p-1} f_{(p,j)} \frac{\partial \tilde{\phi}_{(p)}}{\partial \alpha}(\alpha_i), \quad (1)$$

which is simply the derivative of the continuous analogue to the nodal values along  $\beta = \beta_j$ . However, to show equivalence to the variational form requires a bit of work.

C1584

For simplicity consider a single quadrilateral spectral element with test functions  $\phi_{ij}$  located at nodal points  $(\alpha_i, \beta_j)$ ,  $(i, j) \in [0, \dots, n_p - 1]^2$ . This result is perhaps easiest to show for an arbitrary 2D conservation law,

$$\frac{\partial \psi}{\partial t} + \nabla \cdot F = 0. \quad (2)$$

Using the derivative operator (1) this equation reads

$$\frac{\partial \psi_{ij}}{\partial t} + \frac{1}{J_{ij}} D_\alpha (JF^\alpha) + \frac{1}{J_{ij}} D_\beta (JF^\beta) = 0. \quad (3)$$

Under a variational formulation we have

$$\int \frac{\partial \psi}{\partial t} \phi_{ij} dA + \int \phi_{ij} \nabla \cdot F dA = 0. \quad (4)$$

Using integration by parts,

$$\sum_{m,n} \left( \int \phi_{ij} \phi_{mn} dA \right) \frac{\partial \psi_{mn}}{\partial t} + B - \int \nabla \phi_{ij} \cdot F dA = 0, \quad (5)$$

where  $B$  is the contribution due to the boundary which disappears under direct stiffness summation. Introducing coordinates  $(\alpha, \beta)$  with integration on GLL nodes,

$$\int f dA = \sum_{s=0}^{n_p-1} \sum_{t=0}^{n_p-1} f_{st} J_{st} w_s w_t \Delta \alpha \Delta \beta, \quad (6)$$

where  $w_s$  are the GLL nodal weights from the  $[0, 1]$  reference element. And so the first term reads

$$\begin{aligned} \sum_{m,n} \left( \int \phi_{ij} \phi_{mn} dA \right) \frac{\partial \psi_{mn}}{\partial t} &= \sum_{m,n} (\delta_{i,m} \delta_{j,n} J_{ij} w_i w_j \Delta \alpha \Delta \beta) \frac{\partial \psi_{mn}}{\partial t} \\ &= J_{ij} w_i w_j \Delta \alpha \Delta \beta \frac{\partial \psi_{ij}}{\partial t}. \end{aligned} \quad (7)$$

C1585

For the last term, observe that on a manifold we have

$$\nabla \phi_{ij} \cdot F = g_{pq} F^p \left( g^{qr} \frac{\partial \phi}{\partial x^r} \right) = F^\alpha \frac{\partial \phi}{\partial \alpha} + F^\beta \frac{\partial \phi}{\partial \beta}, \quad (8)$$

and so

$$\int \nabla \phi_{ij} \cdot F dA = \sum_{s=0}^{n_p-1} \sum_{t=0}^{n_p-1} \left[ F^\alpha \frac{\partial \phi_{ij}}{\partial \alpha} + F^\beta \frac{\partial \phi_{ij}}{\partial \beta} \right]_{\alpha=\alpha_s, \beta=\beta_t} J_{st} w_s w_t \Delta \alpha \Delta \beta. \quad (9)$$

But, by construction,

$$\frac{\partial \phi_{ij}}{\partial \alpha} = \frac{\partial \tilde{\phi}_{(i)}}{\partial \alpha} \tilde{\phi}_{(j)}, \quad (10)$$

and  $\tilde{\phi}_{(j)}(\beta_t) = \delta_{jt}$ . This leads to

$$\int \nabla \phi_{ij} \cdot F dA = \left[ \sum_{s=0}^{n_p-1} F_{sj}^\alpha \frac{\partial \tilde{\phi}_{(i)}}{\partial \alpha}(\alpha_s) J_{sj} w_s w_j + \sum_{t=0}^{n_p-1} F_{it}^\beta \frac{\partial \tilde{\phi}_{(j)}}{\partial \beta}(\beta_t) J_{it} w_i w_t \right] \Delta \alpha \Delta \beta. \quad (11)$$

Then in conjunction with (7) this can be written as

$$\frac{\partial \psi_{ij}}{\partial t} - \frac{1}{J_{ij}} \sum_{s=0}^{n_p-1} J_{sj} F_{sj}^\alpha \frac{\partial \tilde{\phi}_{(i)}}{\partial \alpha}(\alpha_s) \frac{w_s}{w_i} - \frac{1}{J_{ij}} \sum_{t=0}^{n_p-1} J_{it} F_{it}^\beta \frac{\partial \tilde{\phi}_{(j)}}{\partial \beta}(\beta_t) \frac{w_t}{w_j} = 0. \quad (12)$$

Equivalence of this equation with (3) follows for a formulation on GLL nodes, where the basis functions satisfy the property

$$\frac{\partial \tilde{\phi}_{(i)}}{\partial \alpha}(\alpha_s) w_s = - \frac{\partial \tilde{\phi}_{(s)}}{\partial \alpha}(\alpha_i) w_i. \quad (13)$$

Note: I do not actually know of a reference for this property and would be very interested to know if one exists.

C1586

*p5150: "A stabilization operator is necessary for finite element methods to avoid dispersive errors associated with spectral ringing." Again, I'm worried about this being cited out of context. In the compatible finite element setting we have produced finite element discretisations that are stabilised purely by the stable vorticity advection operator, with no need for an explicit stabilization operator. We don't need any viscosity or hyperviscosity. It is also the case that DG methods are stabilised purely from the upwinding. Please have a go at narrowing down the language here.*

Agreed. Perhaps "Stabilization is typically needed for co-located (or unstaggered) finite element methods, whether implicitly in the form of upwinding or explicitly in the form of a diffusive operator, to avoid high-frequency dispersive errors associated with spectral ringing."

*p5151: The viscosity operator here feels a bit like you are mixing your drinks in that the non-diffusive part doesn't rely on any test functions, but the viscosity part does. I think this just needs a bit more careful explanation to explain how you are obtaining the operator by dividing by the (diagonal) mass matrix. Is this viscosity operator actually a stable discretisation of the Laplacian i.e. does it have spurious eigenvalues? The flux reconstruction people normally have to resort to LDG/CDG-style operators for this.*

I agree that it's a bit strange to have a non-diffusive operator formulated in differential form and a diffusive operator formulated in variational form. Unfortunately, although using discrete derivatives is a very intuitive way to write the first derivatives in a finite element method (equivalence of the differential and variational form has been demonstrated above), it is not immediately clear how this formulation can be extended for second derivatives. In particular, consider a second-order spectral element method: the second derivative of any linear test function  $\phi$  is zero within an element – but under the variational formulation the second derivative is actually correctly computed. This result extends directly to higher-order methods: A Laplacian-type diffusive operator based on the differential formulation yields spurious eigenvalues (wave modes which are not properly damped). These spurious eigenvalues do not seem to appear for a

C1587

Laplacian formulated using the variational approach (this is definitely true for SE, but admittedly I have not analyzed the DG Laplacian operator described here). Note that the spurious eigenvalues do not appear under the LDG formulation since only first derivatives are used.

It seems, in this case at least, that the mixed drink actually tastes somewhat decent: The differential formulation is used for first derivatives, where it can be used to discretize the non-conservative equations, and the variational formulation is used for the diffusive terms where it appears best suited.

*Please provide a bit more detail on how you obtained the timestep sizes for your numerical calculations, I think this is important as it is the main assessment we can make of computational cost here. It would also be good if you could make some remarks about the relative computational time for one timestep between CG and DG for the same polynomial degree (I know this is tricky since implementation details vary). It's also good to remind the reader how CG and DG DOFs scale with number of elements as a function of polynomial degree (unless I missed this somewhere?).*

For all test cases, time step sizes were chosen to be as large as possible without observing instability over the simulation period. For steady state geostrophically balanced flow (section 6.1),  $dt = 2300s$ ,  $dt = 900s$  and  $dt = 500s$  were found to be unstable for continuous,  $g_2$  discontinuous and  $g_1$  discontinuous elements, respectively ( $ne = 4$ ). For zonal flow over an isolated mountain,  $dt = 520s$ ,  $dt=260s$  and  $dt=130s$  were found to be unstable for continuous,  $g_2$  discontinuous and  $g_1$  discontinuous elements, respectively ( $ne=16$ ). For barotropic instability,  $dt=160s$ ,  $dt=80s$  and  $dt=55s$  were found to be unstable for continuous,  $g_2$  discontinuous and  $g_1$  discontinuous elements, respectively ( $ne=32$ ).

The discontinuous code was not as thoroughly optimized as the continuous code, although based on formulation alone it is clear that the discontinuous code should be more computationally intensive. Overall the discontinuous code was observed to be

C1588

about 30% slower than the continuous code for the same simulation duration and time step size on a low processor count.

On the cubed sphere grid, the discontinuous method has  $6n_e^2n_p^2$  degrees of freedom and the continuous method has  $8 + 8(n_e(n_p - 1) - 1) + 6(n_e(n_p - 1) - 1)^2$  degrees of freedom. In the limit as  $n_e \rightarrow \infty$  this yields a ratio of  $(n_p - 1)^2/n_p^2$  degrees of freedom for the continuous formulation versus the discontinuous formulation. In CAM-SE (I'm not sure about elsewhere), the continuous formulation still stores redundant degrees of freedom in order to reduce computational expense (so this may not be a particularly huge advantage).

Overall: Although I don't want to make an explicit judgement, it seems that the discontinuous formulation isn't worth it.

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