Geosci. Model Dev. Discuss., 7, 1477–1497, 2014 www.geosci-model-dev-discuss.net/7/1477/2014/ doi:10.5194/gmdd-7-1477-2014 © Author(s) 2014. CC Attribution 3.0 License.



This discussion paper is/has been under review for the journal Geoscientific Model Development (GMD). Please refer to the corresponding final paper in GMD if available.

A cusp catastrophe model for alluvial channel regime and classification of channel patterns

Y. Xiao¹, X. J. Shao², and Y. Yang²

¹National Inland Waterway Regulation Engineering Research Center, Chongqing Jiaotong University, Chongqing 400074, China
 ²State Key Laboratory of Hydroscience and Engineering Department of Hydraulic Engineering, Tsinghua University, Beijing 100084, China

Received: 25 December 2013 - Accepted: 15 February 2014 - Published: 24 February 2014

Correspondence to: Y. Xiao (xymttlove@163.com)

Published by Copernicus Publications on behalf of the European Geosciences Union.

Discussion Pa	GN 7, 1477–1	GMDD 7, 1477–1497, 2014						
iper	A cusp ca mo	A cusp catastrophe model						
Discussi	Y. Xia	Y. Xiao et al.						
on P	Title	Title Page						
ape	Abstract	Introduction						
	Conclusions	References						
Discussion Paper	Tables	Figures						
		►I.						
	•	•						
	Back	Close						
—	Full Scr	Full Screen / Esc						
Discussion Paper	Printer-frie Interactive	Printer-friendly Version Interactive Discussion						
	CC U							

Abstract

A cusp catastrophe model for alluvial channel regime is established by selecting suitable parameters to reflect channel stability. An equation is obtained from the equilibrium state of channel regime, which is a cusp catastrophe surface in a translated three

dimensional coordinate. The stability of channel patterns can be identified by such a model in a direct way, and the 2-D projection of the cusp catastrophe surface can be used to classify alluvial channel patterns. Predictions based on this model are consistent with field observations involving about 100 natural rivers. The results indicate that this method may be applied to study the regime of natural rivers and to assist decision
 making in river engineering.

1 Introduction

processes.

Catastrophe theory is a relatively new mathematical method for describing system behavior where a gradually changing force can produce a sudden dramatic effect. ReneThom published the first paper on it in 1968, the first book in 1972 (Thom, 1972), and now it has found applications in a wide variety of disciplines, including physics, biology, psychology, economics, and geology (Stewart and Peregoy, 1983; Gilmore, 1993; Yi, 1995; Hartelman, 1997; Henley, 1976). With the development of nonlinear mathematics, some researchers adopted catastrophe theory to describe the formation of river channels. Thornes (1980) studied the characteristics of Spanish rivers based

on theories of nonlinear mathematics; Richards (1982) brought catastrophe theory into the mechanism of fluvial processes; Graf (1988) described the transformation among straight, meandering and wandering river patterns with the cusp catastrophe theory and pointed out that rivers may result in different patterns although under the same silt-discharge and boundary condition. Nevertheless, the catastrophe theory has just
 explained the phenomenon, and lacked quantitative results on the mechanism of fluvial



This paper established equations for the equilibrium state of channel regime based on the model of cusp-catastrophe surface, the 2-D projection of the cusp catastrophe surface can be used to classify alluvial channel patterns, and the model can provide some useful suggests in practical projects.

5 2 The cusp catastrophe model for alluvial channel regime

2.1 The cusp catastrophe model

10

Of the seven elementary catastrophes in catastrophe theory, the cusp model is the most widely applied. It is often used to model the behavior of a system with two control parameters and a non-linear state parameter. The potential function E for the cusp catastrophe is (Stewart, 1975):

$$E = \frac{1}{4}z^4 + \frac{1}{2}xz^2 - yz \tag{1}$$

Its equilibrium points, as a function of the control parameters x and y, are solutions to the equation as follows:

¹⁵
$$\frac{dE}{dz} = f(x, y, z) = z^3 + xz - y = 0$$

where x, y are defined as the control parameters, and z is the state parameter of the system.

The Cardan's discriminant is written as: $\Delta = 4x^3 + 27y^3$; Eq. (2) Has one solution if $\Delta > 0$, and has three solutions if $\Delta < 0$; The set of values of *x* and *y* for which $\Delta = 0$ demarcates the bifurcation set, and these solutions are depicted as a two dimensional surface living in three dimensional space, the floor of which is the two dimensional (x, y) coordinates system called the "control plane".



(2)

2.2 Determination of the control parameters and the state parameter

The morphology of natural river channels is determined by the interaction of fluid flow with the erodible materials in the channel boundary (Knighton, 1984). The interrelationships between the (1) oncoming bed material load and the sediment carrying capacity

5 of the flow, and (2) the erodibility of the river banks and the erosive power of the water decide essentially the general direction of the self-adjustment. On the basis of the concepts as depicted above, the synthetic indexes of river-bed stability determined by the longitudinal river-bed stability in contrast to the transverse river-bed stability are adopted in the form (Chien Ning, 1958):

10
$$\phi_b = \frac{B}{B_1} = \frac{BJ^{0.2}}{Q^{0.5}}$$

 $\phi_h = \frac{(\gamma_s - \gamma)D_{50}}{\gamma h J}$

here ϕ_h is used as the longitudinal index, ϕ_h is used as the transverse index; h, J are defined as the local water depth and channel slope; Q denotes the bank-full discharge; B is the width of the channel under the bank-full discharge; D_{50} is the median bed 15 material grain size.

The transverse index is a domain factor to describe the channel planforms, this paper decides ϕ_1 as the dominant control parameter, and ϕ_2 as the second control parameter which determines the scouring and deposition of channel beds. The control parameters can be written as:

$$\phi_1 = (\phi_b)^a \tag{5}$$
$$\phi_2 = (\phi_b)^b \tag{6}$$

 $\phi_2 = (\phi_h)$

20

25

where a, b are the weight coefficients, satisfied with a + b = 1. Consider the importance of the oncoming bed material and the erodibility in the river process, we assumed a = b = 1/2 when a + b = 1 in this paper.



(3)

(4)

The state parameter is often used to describe the behavior of a system which is influenced by the control parameters (Stewart, 1975). The sinuosity / is defined as the ratio of channel length to valley length, which reflects the channel planform, responses to the changes of the control parameters. So we choose it as the state parameter in this cusp model.

2.3 The cusp catastrophe model for alluvial channel regime

5

15

Based on the features of topological relationship, the characteristics and the continuity of the equation can be maintained with the coordinate transformation. Translate the coordinate into the $O'(\phi_1, \phi_2, I)$, and obtain the following relationship for control parameters in the form:

$$x = m_1 l + m_2 \phi_1 + m_3 \phi_2 - \phi'_1$$

$$y = n_1 l + n_2 \phi_1 + n_3 \phi_2 - \phi'_2$$

$$z = b_1 l + b_2 \phi_1 + b_3 \phi_2 - l'$$

here ϕ'_1, ϕ'_2, l' is the value of the coordinate in $O'(\phi_1, \phi_2, l)$; b_i, m_i, n_i is the direction cosine between the original and new coordinate system. Substitute Eqs. (5)–(7) to Eq. (2), the equation of the cusp catastrophe under the new coordinate system $O'(\phi_1, \phi_2, l)$ can be expressed as:

$$f(I,\phi_1,\phi_2) = (b_1I + b_2\phi_1 + b_3\phi_2 - I')^3 + (m_1I + m_2\phi_1 + m_3\phi_2 - \phi_1')$$

$$(b_1I + b_2\phi_1 + b_3\phi_2 - I') - (n_1I + n_2\phi_1 + n_3\phi_2 - \phi_2') = 0$$
(8)

As shown in Fig. 1, the rotation around *z* axis of the original coordinate O(x, y, z) to the coordinate system $O'(\phi_1, \phi_2, l)$ contributes to $b_1 = 1$; and b_i, m_i, n_i satisfies the additional condition as:



(7)

$m_1^2 + n_1^2 + b_1^2 = 1$	
$m_2^2 + n_2^2 + b_2^2 = 1$	
$m_3^2 + n_3^2 + b_3^2 = 1$	

ı.

ı.

5

20

The coordinate transformations satisfy the right-handed system:

$$\begin{vmatrix} m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 1$$
(10)

Substitute Eq. (9) to Eq. (10), and obtain:

$$\begin{vmatrix} m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} 0 & \cos \gamma & \sin \gamma \\ 0 & -\sin \gamma & \cos \gamma \\ 1 & 0 & 0 \end{vmatrix}$$
(11)
$$\gamma = \arctan I'$$
(12)

¹⁰ There is no external force when the control parameters ϕ_1, ϕ_2 equal to zero, the channel should maintain its pattern as f(0,0,0) = 0; Eq. (8) can be expressed as:

$$l'^3 - l'\phi_1' - \phi_2' = 0 \tag{13}$$

Substitute Eqs. (11)–(13) into Eq. (8), we have:

$$f(I, \phi_1, \phi_2) = (I - I')^3 + I(\phi_1 \cos \theta - \phi'_1) - \phi_2(\sec \theta - I \cos \theta) + {I'}^3 = 0$$
(14)

$$\theta = \arctan I'$$
(15)

here ϕ'_1 , I' is the critical index, assuming the critical value of state parameter denotes I' = 1 and the critical transverse river-bed stability is $\phi'_1 = 1$ when the rivers attain the equilibrium.

Discussion Paper GMDD 7, 1477-1497, 2014 A cusp catastrophe model Y. Xiao et al. **Discussion** Paper Title Page Abstract Introduction Conclusions References **Figures Discussion** Paper Back Full Screen / Esc Discussion Paper **Printer-friendly Version** Interactive Discussion

(9)

Equation (14) denotes the equilibrium state of the channel regime under the coordinate system $O'(\phi_1, \phi_2, I)$. The surface of equilibrium channel state is formed in 3-dimensional space as shown in Fig. 2: the points located on the upper or lower sheet are minimal (steady equilibrium), a smooth change in (ϕ_1, ϕ_2) at a rate allows the equilibrium to maintain. The middle sheet, often referred as the inaccessible region, represents an unstable maximum (Henley, 1976), if the channels lied in this area, any slight alteration by a control parameter will result in the behavior being transferred to either the upper or lower sheet.

2.4 River data used in the cusp model

- ¹⁰ To investigate the applicability of the cusp model presented herein, the values of the control parameters ϕ_1, ϕ_2 is obtained from a large date set, including date from about 100 natural rivers, which is divided into three types as straight, meandering and braided river; the details of the river data are shown in Table 1. Figure 3 shows the distribution of these rivers in the equilibrium surface.
- As sketched in Fig. 4 one can see 10 points are just on the equilibrium channel state surface (the computed values satisfy Eq. 14). These plot points either on the upper sheet or lower sheet (the global minimum); there are no points in the middle sheet (unstable maximum), this implies that the tendency of adjustment for alluvial channels is the minimum state, not an unstable maximum, the results agrees qualitatively with known channel pattern theories as the minimum entropy (Knighton, 1984). There is no straight river on the equilibrium surface; it demonstrates that the straight pattern should
- not be included as one of the typical patterns that are self-formed (Wang and Ni, 2002). Based on the cusp catastrophe model for alluvial channel regime, some suggestions about the channel stability can be proposed from Fig. 5.
- 1. Choose the section $\phi_2 = 4$ forward and backward about 0.5 units from the equilibrium surface, and the projection of the river positions along the $l - \phi_1$ plane is obtained in Fig. 5. The natural channels are scattered on either side of the



intersection line which defines the control plane into three parts: the meandering, straight and braided rivers belong to the upper, middle and lower sheet in this figure respectively; they may tend to achieve the intersection line by adjusting the control parameters. Straight channels lied on the middle sheet represents an unstable maximum (Arnold, 1994), bifurcation may occur, and result in the meandering or a braided channel pattern without external interference.

5

10

15

20

- 2. Over the short time scale, a channel can adjust its non-fluid boundaries to impose conditions in order to obtain and maintain steady-state equilibrium. Fluctuations occur during the self-adjustment of alluvial channel (Knighton, 1984). From Fig. 5, when the channel is far from the intersection line with high amplitude, the stability of channel can no longer maintain and a cut-off may develop to obtain a new balance; in order to prevent unfavorable impact of river stability change in practice, we should enhance the stability of the channel to control pattern. If the channels are just around the intersection line, the channel pattern can be maintained if the condition of flow discharge and sediment is constant. This implies we can assist some suggestions for river management from the diagram.
- 3. Based on the definition of the state parameter as sinuosity /, the region in the lower sheet is inaccessible (Fig. 5), the channels belong to braided patterns, the width and the beam-depth ratio of these channels are relatively large according to the field data (Table 2).Deviations from equilibrium conditions will trigger the decrease of the control parameter ϕ_2 to attain the available balance with the response between inflowing and outflowing water and sediment discharge, such as decreasing the slope or beam-depth ratio.

3 A cusp catastrophe model for channel pattern classification

²⁵ Numerous descriptive classification schemes have been proposed (Rust, 1978) as well as differences in stability and sedimentation controls in alluvial rivers (Schumm, 1963,



1985). In this section, a new channel pattern classification is presented based on the cusp catastrophe model for alluvial channel regime. The discriminant functions are obtained from the model to distinguish the channel patterns according to the channel stability. The validity of the proposed method is tested by the river data (Table 1).

⁵ The cusp catastrophe model for alluvial channel regime can reflect the stability of channels, one can see the channels achieve to the prescribed minimum of equilibrium state when L = 1; the sinuosity value in excess of 1.5 is often used to discriminate the meandering and braided rivers (Rust, 1978). The cross sections of L = 1 and L = 1.5 are used to divide the equilibrium surface as shown in Fig. 6, and brought into Eq. (15), the discriminators are derived:

When $L = 1: \phi_1 - \phi_2 = 0$ (16) When $L = 1.5: \phi_1 - \phi_2/3 = 0.35$ (17)

The 2-D projection of the natural channel (Fig. 6) along the control $\phi_1 - \phi_2$ plane can ¹⁵ be sketched in Fig. 7. Discriminator (17) represents a reasonable lower bound for wandering channels (the data for wandering channels come from Zhang et al., 1994); Discriminator (18) forms a reasonable upper bound for the meandering channels; braided rivers by the definition used in this analysis, or should at least be regarded as transitional forms based on the dynamic features, and they are in the region between Dis-²⁰ criminators (17) and (18) in this diagram. The control plane is divided into three zones: the upper zone, which can be regarded as the stable zone; the middle zone defined as the transitional stable zone, and the unstable lower zone. It indicates that the classification diagram is in agreement with the known channel pattern theories from the stability feature. Substitute Eqs. (3)–(4) to the Discriminators (17)–(18), discriminant function

²⁵ that predicts the channel pattern can be derived as follows:



Meandering channel:

$$\left(\frac{(\gamma_s - \gamma)D_{50}}{\gamma h J}\right)^{1/2} > 3 \left(\frac{BJ^{0.2}}{Q^{0.5}}\right)^{1/2} - 1.05$$

Braided channel:

5

$$\left(\frac{BJ^{0.2}}{Q^{0.5}}\right)^{1/2} < \left(\frac{(\gamma_s - \gamma)D_{50}}{\gamma h J}\right)^{1/2} < 3\left(\frac{BJ^{0.2}}{Q^{0.5}}\right)^{1/2} - 1.05$$

Wandering channel:

$$\left(\frac{BJ^{0.2}}{Q^{0.5}}\right)^{1/2} < \left(\frac{(\gamma_s - \gamma)D_{50}}{\gamma h J}\right)^{1/2}$$
(20)

As shown in Fig. 7,the channel patterns are difficult to distinguish nearby the origin, according to the catastrophe theory: the region around the origin of the fold line could have remarkably different outcomes of the system behavior. For although the two paths may begin close together their paths are divergent (Arnold, 1994). It is can be used to explain that different planforms may result from the same silt-discharge and boundary condition.

15 4 Conclusion

This paper presents a cusp catastrophe model for alluvial channel regime with suitable parameters to reflect channel stability. An equation is obtained from the equilibrium state of channel regime, which is a cusp catastrophe surface in a translated three dimensional coordinate. The stability of channel patterns can be identified by such



(18)

(19)

a model in a direct way, and the 2-D projection of the cusp catastrophe surface can be used to classify alluvial channel patterns. About 100 natural rivers are used to test the application of the model, and the results indicate that this method may be applied to study the regime of natural rivers and to assist decision making in river management.

⁵ However, some assumptions have proposed before the establishment of the model, further studies are needed to improve understanding of patterning process and to modify the control parameters of the cusp model.

Acknowledgements. This work was supported by the National Natural Science Foundation of China (No. 50879042, 50979043)

10 **References**

20

- Andrews, E. D.: Bed-material entrainment and hydraulic geometry of gravel-bed rivers in Colorado, Bull. Geol. Sot. Am., 95, 371–378, 1984.
- Arnold, V. I.: Dynamical Systems V: Bifurcation Theory and Catastrophe Theory, Springer-Verlag Berlin, Heidelberg, 1994.
- ¹⁵ Chien Ning: On the stability criterion of alluvial streams, Acta Geographica Sinica, 24, 128–142, 1958.

Church, M. and Rood, K.: Catalogue of alluvial river channel regime data, Dep. of Geography, Univ. of British Columbia, Vancouver, 99, 1983.

Gilmore, R.: Catastrophe Theory for Scientists and Engineers, Dover, New York, NY, USA, 1993.

Graf, W. L.: Applications of catastrophe theory in fluvial geomorphology, Modeling Geomorphologic Systems, John Wiley and Sons New York, 33–47, 1988.

Hartelman, P. A. I.: Stochastic Catastrophe Theory, Ph.D. thesis, University of Amsterdam, Amsterdam, the Netherlands, 1997.

Henley, S.: Catastrophe theory models in geology, Math. Geol., 8, 649–655, 1976.

Hey, R. D. and Thome, C. R.: Stable channels with mobile gravel beds, J. Hydraul. Eng., 112, 671–689, 1986.



- Kellerhals, R., Neill, C. R., and Bray, D. I.: Hydraulic and geomorphic characteristics of rivers in Alberta, Alberta Cooperative Research program in Highway and River Eng., Rep 72-I: 52, 1972.
- Knighton, D.: Fluvial Forms and Processes, John Wiley & Sons, Inc., 605 Third Avenue, New York, 1984.
- Lambeek, J. J. P.: Haalbaarheidsstudie Grensmaas, Delft Hydraulics, Report 41437, Emmeloord, the Netherlands, 1994.

5

30

- McCarthy, T. S., Stanistreet, I. G., and Caimcross, B.: The sedimentary dynamics of active fluvial channels on the Okavango fan, Botswana, Sedimentology, 38, 471–487, 1991.
- ¹⁰ Monsalve, G. C. and Silva, E. F.: Characteristics of a natural meandering river in Colombia: Sinu River, in: Meandering, edited by: Elliott, C. M., River Proc. Conf. Rivers, 77–88, 1983.
- Morton, R. A. and Donaldson, A. C.: The Guadelupe River and delta of Texas a modem analogue for some ancient fluvial-deltaic systems, in: Fluvial Sedimentology, edited by: Mi-all, A. D., Can. Sot. Petrol. Geol. Mem., 5, 773–788, 1978b.
- ¹⁵ Mosley, M. P.: Semi-deterministic hydraulic geometry of river channel, South Island, New Zealand, Earth Surf. Process. Land., 6, 127–137, 1981.
 - Richards, K. S.: Rivers: Form and Process in Alluvial Channels, Methuen, London, 1982.
 - Rust, B. R.: A Classification of alluvial channel systems, in Fluvial sedimentology, edited by: Miall, A. D., Memoir No. 5, Cannadian Soc. Petroleum Geologisis, 187–198, 1978.
- Stewart. I.: The seven elementary catastrophes, New Sci., 20, 445–456, 1975. Stewart, I. N. and Peregoy, P. L.: "Catastrophe-Theory Modeling in Psychology", Psychol. Bull., 94, 336–362, 1983.
 - Taylor, G. and Woodyer, K. D.: Bank deposition in suspended load streams, in: Fluvial Sedimentology, edited by: Miall, A. D., Can. Sot. Petrol. Geol. Mem., 5, 257–275, 1978.
- Thom, R.: Stabilité structurelle et morphogénèse: essai d'une théorie générale des modèles, Paris: InterEditions, 351 pp., English translation: Structural stability and morphogenesis; an outline of a general theory of models, with a foreword by: Waddington, C. H. and Benjamin, W. A., Reading, Mass., 1975.
 - Thornes, J. B.: Structural instability and ephemeral channel behavior, Zeitschrift für Geomorphologie Supplement Band, 36, 233–244, 1980.
 - Van der Berg, J. H.: Prediction of alluvial channel pattern of perennial rivers, Geomorphology, 12, 259–279, 1995.



Wang Suiji and Ni Jinren: Straight river: its formation and speciality, J. Geogr. Sci., 12, 72–80, 2002.

Williams, G. P.: Bankfull discharge of rivers, Water Resour. Res., 14, 1141–1154, 1978. Williams, G. P.: River meanders and channel size, J. Hydrol., 88, 147–164, 1986.

5 Yi Chuixiang: Application of Nonlinear Science in Geography, Weather Publication, Beijing, 131–175, 1995.

Zhang Hong-Wu, Jiang En-Hui, and Bai Yong-mei: The Scale Laws of Model Similarity of Hyperconcentrated Flow [M], He Nan Science and Technology Press, Zheng Zhou, 1994.



GMDD 7, 1477-1497, 2014 A cusp catastrophe model Y. Xiao et al. Title Page Abstract Introduction Conclusions References Figures **Tables |**◀ Close Back Full Screen / Esc **Printer-friendly Version** Interactive Discussion

Discussion Paper

Discussion Paper

Discussion Paper

Discussion Paper

Table 1. Observed data of natural rivers.

Data source	No	Width (m)	Discharge (m ³ s ⁻¹)	D ₅₀ (mm)	Slope (×1000)	Depth (m)	Sinuosity (-)
1.Andrews	18	5.21 ~ 83.8	2.21 ~ 255	23 ~ 122	1.74 ~ 22.23	0.29 ~ 1.85	1.07 ~ 1.98
2.Church and Rood	17	5 ~ 104	2.7 ~ 354	33 ~ 89	0.97 ~ 13.7	0.65 ~ 3.06	1.0 ~ 1.65
3.Hey and Thorne	18	6.5 ~ 76.5	3.9 ~ 424	13.9 ~ 83.8	2.108 ~ 11.4	0.68 ~ 3.21	1.33 ~ 2.5
4.Kellerhals et al.	24	18 ~ 442	7.93 ~ 2606	0.2 ~ 145	0.12 ~ 16.5	0.58 ~ 7.2	1.01 ~ 2.2
5.Lambeek	4	111 ~ 450	1841 ~ 5320	12 ~ 73	0.22 ~ 1.6	3.8 ~ 10	1.1 ~ 1.7
6.McCarthy et al.	1	137	630	0.35	0.4	4.1	1.86
Monsalve and Silva	3	140 ~ 160	830 ~ 1200	0.4 ~ 6.2	0.24 ~ 0.8	3~5	1.2 ~ 2.5
8.Morton and Donaldson	2	57 ~ 122	70, 389	0.3,0.45	0.74, 1.01	7,7.6	1.37, 2.24
9.Taylor and Woodyer	1	40	210	0.15	0.05	5.0	2.3
10.Mosley	66	4.2 ~ 1753	1.73 ~ 3112	1.0 ~ 189	0.62 ~ 28.3	0.33 ~ 2.96	1.0 ~ 1.77
11.Williams	4	13.7 ~ 57.9	12.2 ~ 365.3	2.7 ~ 42	1.52 ~ 6.9	0.7 ~ 3.38	1.32 ~ 1.93

(Data sources: 1. Andrews (1984), sinuosity data partly from Church and Rood (1983); 2. Church and Rood (1983); 3. Hey and Thorne (1986); 4. Kellerhals et al. (1972); 5.Lambeek (1994); 6.McCarthy et al. (1991); 7. Monsalve and Silva (1983); 8. Morton and Donaldson (1978b); 9. Taylor and Woodyer (1978); 10. Mosley (1981); 11.Viiliams (1978, 1989))



Fig. 1. The cusp catastrophe model in new coordinate system.



Fig. 2. The equilibrium channel state surface (the dotted line is the intersection between $\phi_2 = 4$ and the equilibrium channel state surface).





Fig. 3. The distribution of natural rivers in the equilibrium channel state surface.





Fig. 4. Rivers on the equilibrium channel surface.











Fig. 6. Classification of alluvial channel patterns based on the cusp catastrophe model (line A is the projection of the intersection line between L = 1.5 and Eq. (15) in the $\phi_1 - \phi_2$ space; line B is the projection of the intersection line between L = 1.0 and Eq. (15) in $\phi_1 - \phi_2$ space).







