

Interactive comment on “Accuracy of the zeroth and second order shallow ice approximation – numerical and theoretical results” by J. Ahlkrona et al.

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Response to general appreciation

The purpose of this paper is to determine the accuracy of the SOSIA, in the form presented in Baral (1999) and Baral et al (2001). We are certain that we have made no algebra errors or coding errors that would change our results significantly. We are sure of this since we thoroughly checked our results and compared our analytical solutions in Eq. 20-23 and A1-A7 with numerical ones, and because our results

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from both our analysis and numerical experiments are conforming with the theory presented in Schoof & Hindmarsh (2009). We perceive the referee comment such that there is a request for a repetition of our numerical experiments with a Newtonian rheology. If this is needed for convincing the reader we are happy to repeat at least some of our numerical experiments (to repeat all of them will be time consuming and perhaps not very interesting) before possible publication in GMD. We have analytical expressions for $t_{xz(0)}$, $v_x(0)$, $t_{xz(2)}$ with $n = 1$, in which the singularities (and the need for σ_{res}) disappear. We also have done numerical experiments indicating that the scaling relations used to derive the SOSIA are accurate for $n = 1$.

Response to major points

1. As mentioned in section 2.3 on line 10-13 on page 4290, we consider the slow sliding case in Schoof & Hindmarsh. We can there add what traction number we used ($\lambda = \epsilon^{1/3}$). In the introduction we can add a couple of sentences about how we will compare our analytical and numerical results to the theory in Schoof & Hindmarsh.
2. We agree that the absence of t_{xx} is indeed disturbing. It is a result of neglecting the boundary layer in the scaling relations. Simply adding t_{xx} to the creep response function does however not fix the problem (yes we tried) but the root of the problem, i.e. the scaling relations need to be addressed. We used SOSIA as it was derived and presented without alternations and showed that indeed the assumption that t_{xx} is of order ϵ^2 everywhere causes errors.
3. See comment above under 'Response to general appreciation'.

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4. The approach by Soucek and Martinec (2008) is an iterative numerical algorithm, that is it takes several iterations for it to converge. The SOSIA was never meant to be an iterative algorithm but a direct computation in the same way as the SIA is. To compare, one can see SOSIA as an iterative process converging in only 2 iterations (first: SIA, second SOSIA), thereby its computational efficiency. To our understanding, due to the different dynamics inside and outside the boundary layer, the only way to construct a higher order model incorporating t_{xx} in the boundary layer without having a system of equations (done iteratively in Soucek) is by matched asymptotics. To our knowledge, there is no complete computer model for practical use based on matched asymptotics.
5. All our experiments and analysis were only for the time-independent case and we have thus no certain answer to this. However, as there are large errors in the SOSIA, we do not believe it is worth the effort to evolve the model in time.
6. We assume pre-conditioner refers to a transformation used to improve the condition number of the problem and thus speed up an iterative solver. Since the SOSIA solution often is very different from the full Stokes solution we believe it is very unlikely that the SOSIA can be used as a pre-conditioner for other higher order problems.. The convergence of a higher order model would be sensitive to the choice of pre-conditioner and choosing another model (here the SOSIA) as pre-conditioner to a problem is often problematic. The SOSIA might be used as an initial guess to a higher order model but the gain in computational work would probably not be significant compared to using the SIA as initial guess instead. We can add this as a comment in the concluding discussion of the paper.

Response to minor points

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1. Thanks for clearing that up for us, we will remove that sentence, and change the sentence at 4289L3.
2. The boundary conditions at the ice surface for the full Stokes case are stated in Eq. 6. The zeroth and second order boundary conditions are not given in the paper but we will include them in section 2.3. They are (for the model problem):

$$\begin{aligned}
 p_{(0)} &= t_{xz(0)}^D = 0 \\
 p_{(2)} &= -t_{zz(0)}^D \\
 t_{xz(2)}^D &= -(t_{zz(0)}^D - t_{xx(0)}^D) \frac{\partial h}{\partial x}
 \end{aligned}$$

$t_{(xx(0))}$ is only singular where $\frac{\partial v}{\partial x}$ is singular, which isn't at the entire ice surface. There is a sentence about this at 4291L10, we can clarify that sentence.

3. The second order creep response function actually never enters the SOSIA since in the expansion procedure the creep response function is Taylor-expanded around $f(\sigma_{(0)})$. On line 4299L18 we are discussing the shear stress $t_{xz(2)}^D$ and there only the zeroth order creep response enters, which is given at 4291L6. However, the product $\sigma_{(0)}\sigma_{(2)}$ does occur in the second order velocity. This product does contain $t_{xx(0)}$. We show below how the second order velocity is derived. In the PhD-thesis of Baral (1999), the second order stress-strain rate relation equation 2.223 gives:

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$$\begin{aligned} \frac{\partial v_{x(2)}}{\partial z} + \frac{\partial v_{z(0)}}{\partial x} &= 2t_{xz(0)}^D \left(A(T'_{(0)}) \frac{df(\sigma_0)}{d\sigma} \sigma_{(2)} + \frac{1}{2} A(T'_{(0)}) \frac{d^2 f(\sigma_0)}{d\sigma^2} \sigma_{(1)}^2 + \right. \\ &\quad \left. \frac{dA(T'_{(0)})}{dT'} f(\sigma_{(0)}) T'_{(2)} + \frac{dA(T'_{(0)})}{dT'} \frac{df(\sigma_0)}{d\sigma} T'_{(1)} \sigma_{(1)} + \frac{1}{2} \frac{d^2 A(T'_{(0)})}{dT'^2} f(\sigma_{(0)}) T'^2_{(1)} \right) + \\ &\quad 2t_{xz(2)}^D A(T'_{(0)}) f(\sigma_{(0)}) + 2t_{xz(1)}^D \left(A(T'_{(0)}) \frac{df(\sigma_0)}{d\sigma} \sigma_1 + \frac{dA(T'_{(0)})}{dT'} f(\sigma_{(0)}) T'_{(1)} \right) \end{aligned}$$

As we do not consider temperature dependence (above T'), and as the first order solution is zero we are left with

$$\frac{\partial v_{x(2)}}{\partial z} + \frac{\partial v_{z(0)}}{\partial x} = 2t_{xz(0)}^D A(T'_{(0)}) \frac{df(\sigma_0)}{d\sigma} \sigma_{(2)} + 2t_{xz(2)}^D A(T'_{(0)}) f(\sigma_{(0)})$$

$$\text{where } \frac{df(\sigma_0)}{d\sigma} = \frac{d\sigma_0^2}{d\sigma} = 2\sigma_0 \frac{d\sigma_0}{d\sigma} = 2\sigma_0 \frac{d(\sigma - \epsilon\sigma_{(1)} - \epsilon^2\sigma_{(2)} - \dots)}{d\sigma} \approx 2\sigma_0$$

Inserting the above expression and integrating using the boundary conditions stating that both the zeroth and second order velocity is zero at the base gives

$$\begin{aligned} v_{x(2)} &= -\frac{\partial}{\partial x} \int_{b(0)}^z v_{z(0)} dz' + 4 \int_{b(0)}^z t_{xz(0)}^D A(T'_{(0)}) \sigma_{(0)} \sigma_{(2)} dz' + \\ &\quad 2 \int_{b(0)}^z t_{xz(2)}^D A(T'_{(0)}) f(\sigma_{(0)}) dz'. \end{aligned}$$

From Baral et al. (2001) Eq. 5.94 (in 2D with the first order solution set to zero) we have that

$$2\sigma_{(0)}\sigma_{(2)} = 2t_{xz(0)}^D t_{xz(2)}^D + (t_{xx(0)}^D)^2$$

Inserting this into the expression for $v_{x(2)}$ we finally get Eq. 19 in the paper.

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4. To our knowledge there is no computer model using the dual expansions of Schoof & Hindmarsh, even for fast sliding.
5. Our perception is that boundary layers are always assumed to be thin. We will rephrase the sentence to something like "In fluid dynamics boundary layers are usually assumed to be thin, but as found in Ahlkrone et al. (2013) it is ..." The boundary layer thickness will *not* be equal to $\epsilon^{1/3}$, rather it is $O(\epsilon^{1/3})$, which only tells us how the thickness varies with ϵ , nothing about its absolute thickness. In another publication (<http://qjmam.oxfordjournals.org/content/early/2013/08/09/qjmam.hbt009.short>) we studied the boundary layer thickness and found it to be thick for ϵ that could be expected for ice sheets.
6. True, we will change the sentence. For the ice surface (in the model problem) to be $O(1)$ the angle we need $\tan(\alpha) = \epsilon$. This can be seen from non-dimensionalizing the surface h : $\tilde{h} = \frac{h}{[H]} = \frac{-[L]\tilde{x}\tan(\alpha)}{[H]} = \frac{-\tilde{x}\tan(\alpha)}{\epsilon}$.
7. Thank you, we will change that.
8. Yes, in Eq. 5.
9. We are not entirely sure we understand the question. The singularities come from the creep response being zero at the surface, which it will be in any coordinate system. Do you mean that there would be numerical difficulties with having the coordinate system aligned with gravity and not slope?

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10. We ourselves doubted our solutions for a long time and thus checked, double-checked and triple-checked them. We have compared our analytical solutions to our numerical ones thoroughly and the analytical solutions were actually derived partly using the symbolic algebra functionality in Matlab. It should be possible to check the momentum balance using a symbolic algebra program, which we can do if necessary to convince GMD-readers.
11. It is true that we don't consider time-evolution, but there can be oscillations in the solutions if a staggered grid is not used. This is a tupe of instability.
12. It is true that we do not know how other full Stokes codes treats this, we will clarify the sentence.
13. For the SIA we use all ϵ , but not for SOSIA. We will clarify this.
14. At 4309L10 we do not mean that SOSIA would work, just that it would be convenient if it did since it is computationally inexpensive. We will clarify this in the paper. As mentioned in the following lines there would be problems for high aspect ratios. As there are fast changes in the ice dynamics near the grounding line we expect that the effective aspect ratio in that area would be large. Also we do of course not expect the SOSIA to work at ice shelves.

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