An Improved Non-Iterative Surface Layer Flux Scheme for Atmospheric Stable Stratification Condition

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Abstract

Parameterization of turbulent fluxes under stably stratified conditions has always been a challenge. Current surface fluxes calculation schemes either need iterations or suffer low accuracy. In this paper, a non-iteration scheme is proposed to approach the classic iterative computation results using multiple regressions. It can be applied to the full range of roughness status $10 \le z/z_0 \le 10^5$ and $-0.5 \le \log(z_0/z_{0h}) \le 30$ under stable conditions $0 < Ri_B \le 2.5$. The maximum (average) relative errors for the turbulent transfer coefficients for momentum and sensible heat are 12% (1%) and 9% (1%), respectively.

1 Introduction

In weather or climate models, the earth's surface is the boundary that needs to be resolved physically (Chen and Dudhia, 2001). The condition of atmosphere aloft (e.g., wind, temperature and humidity) is highly dependent on the momentum, sensible heat and latent heat fluxes at surface. Currently, the exchanges of momentum and heat between the earth's surface and the atmosphere are usually calculated with various schemes based on Monin-Obukhov similarity theory (hereinafter MOST, Monin and Obukhov, 1954) in models. These schemes (e.g., Paulson, 1970; Businger, 1971; Dyer 1974; Holtslag and De Bruin, 1988, Beljaars and Holtslag, 1991; Janjić 1994; Launiainen, 1995; Högström, 1996) are similar to each other but the differences among them exist due to different observational data and/or mathematical solutions that were used in retrieving the schemes. One commonly used scheme is Businger-Dyer (BD) equation (Businger, 1966; Dyer, 1967). However, the BD equation suppresses fluxes under stable condition too quickly and is not applicable when the Richardson number exceeds a critical value (Louis, 1979). Holtslag and De Bruin (1988) and Beljaars and Holtslag (1991) proposed alternative schemes which can be used under very stable conditions. With data collected in the field program CASES-99 (Cooperative Atmosphere-Surface Exchange Study-99) (Poulos et al., 2002), Cheng and Brutsaert (2005, CB05 hereinafter) further provided a new scheme and it is confirmed to perform better by later research (Guo and Zhang, 2007; Jiménez et al, 2012). Based on the measurements made during experiment SHEBA in Arctic and Halley 2003 experiment in Antarctica, Grachev et al. (2007) and Sanz Rodrigo and Anderson (2013) proposed different similarity functions, respectively. Through systematic mathematical analysis, Sharan and Kumar (2011) proved that similarity functions of CB05 and Grachev et al. (2007) were applicable in the whole stable stratification region. However, all of these studies are based on MOST and application of MOST in very stable condition is in doubt since it assumes that turbulence is continuous and stationary, while in very stable condition turbulence is weak, sporadic and patchy (Sharan and Kumar, 2011). Grachev et al. (2013) indicates that the applicability of local MOST in stable conditions is limited by the inequalities, when both gradient and flux Richardson numbers are below their "critical values" about 0.20-0.25. Further, MOST predicts that mean gradients of turbulence become independent of z in very stable condition, Wyngaard and Coté (1972) first referred to this limit as ‘z-less stratification’. BD equations follow this prediction, but CB05 and Grachev et al. (2007) do not. To avoid these holdbacks and self-correlation of MOST, Sorbjan (2010) and Sorbjan and Grachev (2010) discussed an alternative local scaling for the stable boundary layer (referred to as gradient-based scaling) when different universal functions plotted versus the gradient Richardson number instead of the Monin-Obukhov stability parameter.

Another critical issue regarding the fluxes calculation with MOST is the numerical iteration. Under unstable condition, the iteration normally converges within 5 steps (Fairall et al, 1996). By taking advantage of a bulk Richardson number parameterization for an improved first guess

(Grachev and Fairall, 1997), the iteration can be reduced to 3 steps (Fairall et al, 2003). In the Weather Research Forecasting (WRF) model (Skamarock et al., 2008) MM5 similarity surface module, the flux variables from the previous time step are used to calculate the fluxes at current time step and such an approach can yield reasonable result (Jiménez et al, 2012). On the other hand, under stable condition, the flux calculation takes many more steps to converge and hence is time consuming. To avoid the iteration process, a series of non-iterative schemes are proposed (e.g., Loius, 1979, Garratt, 1992, Launiainen, 1995, Song, 1998, De Bruinet al. 2000, Yang et al., 2001; Li et al., 2010), but they all fail to cover the full range of $-0.5 \le kB^{-1} \le 30$, $10 \le z/z_0 \le 10^5$ and $-5.0 \le Ri_B \le 2.5$, which is pointed out by Wouters et al. (2012, WRL12 hereinafter). Here $kB^{-1} = \ln(z_0/z_{0h})$. z is the reference height; and z_0 and z_{0h} are the aerodynamic and thermal roughness lengths, respectively. Ri_B is the bulk Richardson number. Following WRL12, the condition that $Ri_B > 2.5$ is not considered in this study, because it represents extremely stable stratification with very weak wind and little flux exchange. To calculate fluxes under all conditions, and also to include the roughness sublayer effect, WRL12 proposed an updated scheme based on the iterated results of CB05 under stable condition. However, for a given Ri_B, WRL12 uses only one equation to cover the whole large range of z/z_0 and kB^{-1}, which results in biases at some z/z_0 and kB^{-1} conditions. Therefore, to avoid the iteration process and keep the accuracy at the same time, the objective of this paper is to propose a group of equations that divide the calculation into 8 regions according to z_0 and z_{0h} values. To compare with WRL12, and with the fact that CB05 equations are currently widely accepted, the new equations are also based on the iterated results of CB05 equations. Section 2 describes the calculation results from CB05 and WRL12. Section 3 introduces the new equations, and section 4 intercompares these schemes. Summary and conclusions are presented in section 5

2 Revisiting CB05 and WRL12

The momentum flux τ and sensible heat flux H are defined as:

$$\tau \equiv \rho u_*^2 \quad (1)$$

$$H \equiv -\rho c_p u_* \theta_* \quad (2)$$

Here u_* is the friction velocity, θ_* is the temperature scale, ρ the air density and c_p the specific heat capacity at constant pressure. Based on MOST, the friction velocity u_* and temperature scale θ_* can be calculated by:

$$u_* = uk / [\ln(\frac{z}{z_0}) - \psi_{\mathrm{m}}(\frac{z}{L}) + \psi_{\mathrm{m}}(\frac{z_0}{L}) + \psi_{\mathrm{m}}^*(\frac{z}{L}, \frac{z}{z_*})] \quad (3)$$

$$\theta_* = (\theta - \theta_0)k / [\ln(\frac{z}{z_{0\mathrm{h}}}) - \psi_{\mathrm{h}}(\frac{z}{L}) + \psi_{\mathrm{h}}(\frac{z_{0\mathrm{h}}}{L}) + \psi_{\mathrm{h}}^*(\frac{z}{L}, \frac{z}{z_*})] \quad (4)$$

Here u and θ are the wind speed and potential temperature at the reference height z. k is the von Karman constant. z_* is the roughness sublayer height. θ_0 is the potential temperature at the height of $z_{0\mathrm{h}}$. ψ_{m} and ψ_{h} are the integrated stability functions for momentum and heat, respectively. ψ_{m}^* and ψ_{h}^* are the correction functions accounting for roughness sublayer effect. L is the Obukhov length defined as：

$$L \equiv u_*^2 \overline{\theta} / (kg\theta_*) \quad (5)$$

ψ_{m}^* and ψ_{h}^* are given by De Ridder (2010):

$$\psi_{\mathrm{m,h}}^*(\frac{z}{L}, \frac{z}{z_*}) = \int_z^{\infty} \frac{\phi_{\mathrm{m,h}}(\frac{z'}{L})}{z'} e^{-\mu_{\mathrm{m,h}} \frac{z'}{z_*}} dz' \quad (6)$$

$\mu_{\mathrm{m}} = 2.59$, $\mu_{\mathrm{h}} = 0.95$, and $\phi_{\mathrm{m,h}}$ are the stability functions for momentum and heat. Following Sarkar and De Ridder (2010) and WRL12, $z_* / z_0 = 16.7$ is adopted in this study.

CB05 gives the form of $\phi_{\mathrm{m,h}}$ and $\psi_{\mathrm{m,h}}$:

$$\phi_{\mathrm{m}} = 1 + a \frac{\zeta + \zeta^b (1+\zeta^b)^{\frac{1-b}{b}}}{\zeta + (1+\zeta^b)^{\frac{1}{b}}} \quad (7)$$

$$\phi_{\mathrm{h}} = 1 + \mathrm{c} \frac{\zeta + \zeta^d (1+\zeta^d)^{\frac{1-d}{d}}}{\zeta + (1+\zeta^d)^{\frac{1}{d}}} \quad (8)$$

$$\psi_{\mathrm{m}} = -a \ln(\zeta + (1+\zeta^b)^{\frac{1}{b}}) \quad (9)$$

$$\psi_{\mathrm{h}} = -c\ln(\zeta + (1+\zeta^{d})^{\frac{1}{d}}) \quad (10)$$

Here $a = 6.1, b = 2.5, c = 5.3$ and $d = 1.1$. $\zeta \equiv z/L$ is the stability parameter.

With Eqs. (3), (4), (5), and (6), $\phi_{\mathrm{m,h}}$ and $\psi_{\mathrm{m,h}}$ of CB05, fluxes can be calculated through iterations: with a first guess of ζ, u_* and θ_* can be calculated from Eqs. (3) and (4), then ζ again can be derived from Eq. (5).This procedure iterates until the results converge. The relationships of $\zeta \sim Ri_{\mathrm{B}}$, $\zeta \sim \ln(z/z_0)$,and $\zeta \sim \ln(z_0/z_{0\mathrm{h}})$ from CB05 are shown in Fig. 1, Fig. 2 and Fig. 3, respectively. Conditions with $Ri_B = 0.05, 0.2, 0.5$, $z/z_0 = 10, 1000, 10^5$ and $kB^{-1} = -0.5, 15, 30$ are plotted. However, due to the limitation of computational time in numerical weather and climate models, the calculation results after 5 steps are always taken to approximate the fluxes (e.g., MYJ and MYNN surface module in WRF model, Janjić, 1996, Nakanishi M, Niino, 2006). It is found that with the first guess of $\zeta_0 = Ri_{\mathrm{B}} \dfrac{[\ln(z/z_0)]^2}{\ln(z/z_{0\mathrm{h}})}$ and 5 steps of iteration, the results are still far away from the precise value. Fig. 4 presents the relative error $\Delta\zeta$ for various Ri_B with $z/z_0 = 10, 1000, 10^5$ and $kB^{-1} = -0.5, 15, 30$. The relative error $\Delta\zeta$ that is calculated by Eq. 11can exceed 70% under certain conditions.

$$\Delta\zeta = \begin{cases} \dfrac{\left|\zeta_{(\mathrm{cal})} - \zeta_{(\mathrm{precise})}\right|}{\zeta_{(\mathrm{precise})}} \times 100\%, \text{ for } \left|\zeta_{(\mathrm{cal})} - \zeta_{(\mathrm{precise})}\right| \geq 0.01 \\ 0, \text{ for } \left|\zeta_{(\mathrm{cal})} - \zeta_{(\mathrm{precise})}\right| < 0.01 \end{cases} \quad (11)$$

where $\zeta_{(\mathrm{cal})}$ is the calculation result, and $\zeta_{(\mathrm{precise})}$ is the precise result from the ultimate iteration of CB05 (when $|\zeta_{(\mathrm{n+1})} - \zeta_{(\mathrm{n})}| < 0.1\%\zeta_{(\mathrm{n})}$, $\zeta_{(\mathrm{n})}$ is adopted as $\zeta_{(\mathrm{precise})}$, and here n indicates the iteration step). Fig. 5 shows the steps needed to converge into 5% relative error with CB05 equations for various Ri_B with $z/z_0 = 10, 1000, 10^5$ and $kB^{-1} = -0.5, 15, 30$. It shows that when $Ri_B = 0.74$, $z/z_0 = 10$ and $kB^{-1} = 30$, more than 80 steps of iteration are needed to reduce the calculation error within 5%. The iteration takes more steps to converge when there is a larger aerodynamic roughness length z_0 and a smaller thermal roughness length $z_{0\mathrm{h}}$, which is common over an urban surface (Sugawara and Narita, 2009). When $z/z_0 = 10$ and $kB^{-1} = 30$, the largest error can reach 75% after 5 steps iteration (Fig. 4) and 82 steps are needed for the results to converge (Fig. 5). However, when z/z_0 becomes large, for example $z/z_0 = 10^5$ (i.e., a representative value for a smooth sea surface), 5 steps are enough for the results to be within 5% error under all kB^{-1} and Ri_{B} conditions (Fig. 5).

To avoid the iteration, and based on CB05's iteration results, WRL12 proposed the following set of equations:

$$\begin{aligned}\zeta_t = &-0.316 - 0.515e^{-L_{0\mathrm{H}}} + 25.8e^{-2L_{0\mathrm{H}}} + 4.36{L_{0\mathrm{H}}}^{-1} - 6.39{L_{0\mathrm{H}}}^{-2} \\ &+ 0.834\log(L_{0\mathrm{M}}) - 0.0267\log^2(L_{0\mathrm{M}})\end{aligned}, \quad (12)$$

$$Ri_{\mathrm{B,t}} = \zeta_{\mathrm{t}} \frac{L^*_{0\mathrm{H}} + S^*_{0\mathrm{H}}\beta_{\mathrm{H}}\zeta_{\mathrm{t}}}{(L^*_{0\mathrm{M}} + S^*_{0\mathrm{M}}\beta_{\mathrm{M}}\zeta_{\mathrm{t}})^2}, \quad (13)$$

$$\zeta = \frac{-L^*_{0\mathrm{M}}}{S^*_{0\mathrm{M}}\beta_{\mathrm{M}}} - \frac{BC}{4(S^*_{0\mathrm{M}}\beta_{\mathrm{M}})^3(B^2+|Cr|)} + \frac{B - \sqrt{B^2+Cr} + \dfrac{BCr}{2(B^2+|Cr|)}}{2(S^*_{0\mathrm{M}}\beta_{\mathrm{M}})^3 r}, \text{ (for } Ri_{\mathrm{B}} \le Ri_{\mathrm{B,t}}\text{)}, \quad (14)$$

$$\zeta = \zeta_{\mathrm{t}} + D(\zeta_{\mathrm{t}})(Ri_{\mathrm{B}} - Ri_{\mathrm{B,t}}), \text{ (for } Ri_{\mathrm{B}} > Ri_{\mathrm{B,t}}\text{)}, \quad (15)$$

$$D(\zeta_{\mathrm{t}}) = \frac{(L^*_{0\mathrm{M}} + S^*_{0\mathrm{M}}\beta_{\mathrm{M}}\zeta_{\mathrm{t}})^3}{L^*_{0\mathrm{M}}L^*_{0\mathrm{H}} + \zeta_{\mathrm{t}}(2S^*_{0\mathrm{H}}\beta_{\mathrm{H}}L^*_{0\mathrm{M}} - S^*_{0\mathrm{M}}\beta_{\mathrm{M}}L^*_{0H})}, \quad (16)$$

where

$$L_{0\mathrm{i}} = \ln(z/z_{0\mathrm{i}}), \text{ (i stands for M or H)}, \quad (17)$$

$$L^*_{0\mathrm{i}} = L_{0\mathrm{i}} + \frac{1}{\lambda}\ln\left(1 + \frac{\lambda}{\mu_{\mathrm{i}}\frac{z}{z_*}}\right)e^{-\mu_{\mathrm{i}}\frac{z}{z_*}}, \text{ (i stands for M or H)}, \quad (18)$$

$$r = Ri_{\mathrm{B}} - S^*_{0\mathrm{H}}\beta_{\mathrm{H}} / (S^*_{0\mathrm{M}}\beta_{\mathrm{M}})^2 \quad (19)$$

$$B = S^*_{0\mathrm{M}}\beta_{\mathrm{M}}L^*_{0\mathrm{H}} - 2S^*_{0\mathrm{H}}\beta_{\mathrm{H}}L^*_{0\mathrm{M}} \quad (20)$$

$$C = 4(S^*_{0\mathrm{M}}\beta_{\mathrm{M}})^2 L^*_{0\mathrm{M}}(S^*_{0\mathrm{H}}\beta_{\mathrm{H}}L^*_{0\mathrm{M}} - S^*_{0\mathrm{M}}\beta_{\mathrm{M}}L^*_{0\mathrm{H}}) \quad (21)$$

$$S^*_{0\mathrm{i}} = 1 - z_{0\mathrm{i}}/z + \left(1 + \frac{\nu}{\mu_{\mathrm{i}}\frac{z}{z_*}}\right)\frac{1}{\lambda}\ln\left(1 + \frac{\lambda}{\mu_{\mathrm{i}}\frac{z}{z_*}}\right)e^{-\mu_{\mathrm{i}}\frac{z}{z_*}} \quad (22)$$

where $\lambda = 1.5$, $\nu = 0.5$, β_{M}=4.76+7.03$z_0/z + 0.24z_{0\mathrm{h}}/z_0$ and β_{H}=5. First, $Ri_{\mathrm{B,t}}$ is calculated from Eqs. (12) and (13), and then ζ can be derived from Eqs. (14) or (15). Fig. 6 presents the relative error of ζ with WRL12 equations compared with iterated results of CB05 for various Ri_B with $z/z_0 = 10, 1000, 10^5$ and $kB^{-1} = -0.5, 15, 30$. It shows that the relative error of WRL12 exceeds 20% when Ri_B is small, and exceeds 50% when Ri_B becomes large.

3 Derivation of the new scheme

It can be seen from Fig. 1, Fig. 2 and Fig. 3 that ζ varies with Ri_B, $\log(z/z_0)$ and kB^{-1} with remarkable nonlinearity. Specially, when kB^{-1} is large, $\zeta \sim z_0$ relationship can hardly be approximated by a cubic equation at some Ri_B values (Fig. 2).Correspondingly, when z_0 is large, $\zeta \sim z_{0h}$ also needs a high power series equation to approximate (at least cubic fit is not enough, Fig. 3). In order to reduce the complexity, weakly and strongly stable conditions are treated separately in previous studies (e.g., Launiainen, 1995; Li et al., 2010; WRL12). Analogously, multiple regions are considered for z_0 and z_{0h} for the regression of $\zeta = f\left(Ri_B, L_{0M}, kB^{-1}\right)$ in this paper. In this way, the complexity of the equations can be reduced and at the same time their accuracy can be maintained. Although the total number of equations is increased due to the division of z_0 and z_{0h}, the calculation efficiency is still enhanced since the logical judgment of the region according to z_0 and z_{0h} values in programme codes takes much less time than iterations. The critical issue here is how to divide the z_0 and z_{0h} regions in a reasonable way to obtain the smallest number of regions but the highest accuracy. For this purpose, the z_0 and z_{0h} are first divided into 13 and 14 sections according to the values of z/z_0 and z_0/z_{0h}, respectively. For z/z_0, the sections are 10~20, 20~40, 40~80, …, 10240~20480, 20480~40960 and 40960~10^5; for z_0/z_{0h}, the sections are 0.607~1, 1~10, 10~100, 100~10^3, 10^3~10^4, …, 10^{11}~10^{12} and 10^{12}~1.07×10^{13}. $z/z_0 \in 10 \sim 20$ and $z_0/z_{0h} \in 10^{12} \sim 1.07\times10^{13}$ is the region that needs the highest power series equation to approximate. This region is firstly chosen to find a maximum critical value of ζ_{c1} that can make the regression:

$$\zeta = f\left(Ri_B, L_{0M}, kB^{-1}\right) = Ri_B \sum C_{ijk}\ Ri_B^i\ L_{0M}^j\ (L_{0H} - L_{0M})^k \qquad (23)$$

be within 5% error when $\zeta \in 0 \sim \zeta_{c1}$. Here i, j, and k = 0, 1, 2, and 3, and $i + j + k \le 4$. C_{ijk} are the coefficients from regression. It is found that $\zeta_{c1} = 0.33$ meets this criterion. Then some of the z_0 and z_{0h} regions can be merged with each other for the section $\zeta \in 0 \sim 0.33$ and a total of

8 $z_0 - z_{0h}$ regions are left in the $z_0 - z_{0h}$ plane. In other words, the regression error of Eq. (23) can be kept within 5% in any of the 8 regions when $\zeta \in 0 \sim 0.33$ (Table 1). Thus, for these 8 regions, it can be found that with the sections divided by the specified critical values ζ_{cp} (where p is 1,2,3, … it indicates the section and its maximum value depends on the $z_0 - z_{0h}$ region), the regression error with Eq. (23) can be kept within 5% for $\zeta \le 0.5$ and 10% or $\zeta > 0.5$. For a given pair of z_0 and z_{0h}, the division by ζ_{cp} can be transformed to Ri_{Bcp}:

$$Ri_{Bcp} = \sum C_{mn} \ \log^m(L_{0M}) \ (L_{0H} - L_{0M})^n \tag{24}$$

Here m, n = 0, 1, 2, and $m+n \le 3$; p is 1,2,3, …, which indicates the section and its maximum value depends on the $z_0 - z_{0h}$ region. For region 1 and 7, the maximum p is 6, while for other regions it varies between 3 and 5. The coefficients for Eq. (24) are shown in Table 2. The Ri_{Bcp} then cut the 0-2.5 Ri_B range into several sections: section 1 is from 0 to Ri_{Bc1}, section 2 from Ri_{Bc1} to Ri_{Bc2}, and so on. The coefficients for Eq. (23) in each section are given in Tables 3~10.

The procedure to obtain these coefficients are summarized below:

1) Divide z / z_0 into 13 sections: 10~20, 20~40, 40~80, …, 10240~20480, 20480~40960 and 40960~10^5; divide z_0 / z_{0h} in to 14 sections: 0.607~1, 1~10, 10~100, 100~10^3, 10^3~10^4, … , 10^{11}~10^{12} and 10^{12}~1.07×10^{13}.

2) Use the region $z / z_0 \in 10 \sim 20$ and $z_0 / z_{0h} \in 10^{12} \sim 1.07\times10^{13}$ to find ζ_{c1}.
   Method: when $\zeta \in 0 \sim \zeta_{c1}$, regression with Eq. (23) is kept within 5% error.

   Result: $\zeta_{c1} = 0.33$ found.

3) Use $\zeta_{c1} = 0.33$ to recombine z / z_0 and z_0 / z_{0h} sections defined in step 1.
   Method: Variations of combinations of the 13 sections of z / z_0 and 14 sections of z_0 / z_{0h} are tested to minimize the numbers of regions, and regression with Eq. (23) and $\zeta \in 0 \sim 0.33$ is kept within 5% error.

   Result: 8 regions found (Table 1)

4) For each of the 8 regions, find ζ_{c1}, ζ_{c2}, …, ζ_{cp}, …
   Method: when $\zeta \in 0 \sim \zeta_{c1}$, or $\zeta_{c1} \sim \zeta_{c2}$, …, or $\zeta_{c(p-1)} \sim \zeta_{cp}$, …, regression with Eq. (23) is kept within 5% error for $\zeta \le 0.5$ and 10% error for $\zeta > 0.5$.

Result: $\zeta_{c1}, \zeta_{c2}, \ldots, \zeta_{cp}, \ldots$, for each region found

5) Transfer $\zeta_{c1}, \zeta_{c2}, \ldots, \zeta_{cp}, \ldots$, to $Ri_{Bc1}, Ri_{Bc2}, \ldots, Ri_{Bcp}, \ldots$, with Eq. (24)

Method: for each region, when $Ri_{B} \in 0 \sim Ri_{Bc1}$, or $Ri_{Bc1} \sim Ri_{Bc2}$, …, or $Ri_{Bc(p-1)} \sim Ri_{Bcp}$, …,regression with Eq. (23) is kept within 5% error for $\zeta \le 0.5$ and10% error for $\zeta > 0.5$.

Result: coefficients of Eq. (23) and Eq. (24) are derived.

The calculation procedure for a given group of z_0, z_{0h} and Ri_B is that: 1) find the region according to z_0 and z_{0h} with Table 1; 2) Find the section according to the region and Ri_B with Eq. (24) and coefficients in Table 2; and 3) In Table 3-10 find the coefficients for the particular region and section and use Eq. (23) to calculate ζ. Fig. 7 presents the relative error of ζ with new equations compared with iterated results of CB05 for various Ri_B with $z/z_0 = 10, 1000, 10^5$ and $kB^{-1} = -0.5, 15, 30$. With the new equations, the relative error is controlled to be within 10% for the whole range. Specially, when $\zeta \le 0.5$, the relative error is within 5% since it happens more often in the real conditions (Fig. 8).

4 Comparison of the results from CB05 with 5 steps iteration, WRL12 and the new scheme

The maximum and average relative error of ζ, C_M and C_H calculated from CB05 with 5 steps iteration, WRL12 and the new scheme are shown in Fig. 8, Fig. 9 and Fig. 10 for various ζ with $z/z_0 = 10, 1000, 10^5$ and $kB^{-1} = -0.5, 15, 30$. C_M and C_H are the transfer coefficients for momentum and sensible heat respectively, and:

$$C_M = \frac{k^2}{[\ln(\frac{z}{z_0}) - \psi_m(\zeta) + \psi_m(\frac{z_0}{z}\zeta) + \psi_m^*(\zeta, \frac{z}{z_*})]^2} \quad (25)$$

$$C_H = \frac{k^2}{[\ln(\frac{z}{z_0}) - \psi_m(\zeta) + \psi_m(\frac{z_0}{z}\zeta) + \psi_m^*(\zeta, \frac{z}{z_*})][\ln(\frac{z}{z_{0h}}) - \psi_h(\zeta) + \psi_h(\frac{z_0}{z}\zeta) + \psi_h^*(\zeta, \frac{z}{z_*})]} \quad (26)$$

To speed up the calculation, $\psi^*_{\mathrm{m,h}}(\zeta, \frac{z}{z_*})$ is not calculated from Eq. (6) but rather from the non-integral equation proposed by De Ridder (2010):

$$\psi^*_{\mathrm{m,h}}(\zeta, \frac{z}{z_*}) = \phi_{\mathrm{m,h}}[(1+\frac{\nu}{\mu z / z_*})\zeta]\frac{1}{\lambda}\ln(1+\frac{\lambda}{\mu z / z_*})\exp(-\mu z / z_*) \tag{27}$$

where $\lambda = 1.5$, $\mu = \mu_{\mathrm{m}} = 2.59$, $\mu = \mu_{\mathrm{h}} = 0.95$ and $\nu = 0.5$. The relative error for C_{M} and C_{H} is calculated from:

$$\Delta C_{\mathrm{M,H}} = \frac{\left|C_{\mathrm{M,H(cal)}} - C_{\mathrm{M,H(precise)}}\right|}{C_{\mathrm{M,H(precise)}}} \times 100\% \tag{28}$$

where $C_{\mathrm{M,H(cal)}}$ is calculated with $\zeta_{\mathrm{(cal)}}$ from the three different methods, and $C_{\mathrm{M,H(precise)}}$ is calculated with $\zeta_{\mathrm{(precise)}}$ from the ultimate iteration of CB05.

Maximum error indicates the maximum error for a particular ζ under various z_0 and $z_{0\mathrm{h}}$ conditions, while average error is calculated from

$$AverageError(\zeta) = \frac{\int_{-0.5}^{30}\int_{\log(10)}^{\log(10^5)} Error(\zeta)\ d\log(\frac{z}{z_0})\ d\log(\frac{z_0}{z_{0\mathrm{h}}})}{\int_{-0.5}^{30}\int_{\log(10)}^{\log(10^5)} d\log(\frac{z}{z_0})\ d\log(\frac{z_0}{z_{0\mathrm{h}}})} \tag{29}$$

Here $Error(\zeta)$ indicates $\Delta\zeta$ or $\Delta C_{\mathrm{M,H}}$ at a particular ζ, z_0 and $z_{0\mathrm{h}}$. Although Eq. (28) presents the form of continuous integral, it is actually calculated discretely with interval 0.035 for $\log(\frac{z}{z_0})$ and 0.1 for $\log(\frac{z_0}{z_{0\mathrm{h}}})$.

The results indicate that the maximum $\Delta\zeta$ exceeds 50% when using CB05 with 5 steps iteration or WRL12, while the averaged $\Delta\zeta$ for the two methods both exceeds 15%. On the contrary, the maximum $\Delta\zeta$ of the new scheme is always smaller than 5% (when $\zeta \le 0.5$) and 10% (when $\zeta > 0.5$), and the average $\Delta\zeta$ is always smaller than 2% in the whole range. The maximum ΔC_{M} from CB05 with 5 steps iteration (WRL12) exceeds 50% (40%), and average ΔC_{M} exceeds 30% (8%). The maximum ΔC_{H} from CB05 with 5 steps iteration (WRL12) exceeds 50% (24%), and average ΔC_{H} exceeds 18% (6%). Comparatively, the new scheme controls the maximum ΔC_{M} (ΔC_{H}) to be within 12% (9%) and the average ΔC_{M} (ΔC_{H}) within 1% (1%). Table 11 summarizes the characteristics of the four methods.

5 Summary and conclusions

Although CB05 provides a way to calculate surface fluxes under stable condition, its practical usage is confined due to the involved iteration process. It has been shown that iteration with 5 steps will result in large calculation errors, especially when z/z_0 is small and kB^{-1} is large, which is common over an urban surface. WRL12 proposed a way to avoid the iteration, but it introduces large error in the calculation procedure so that its calculation accuracy needs be improved. Through dividing the $z_0 - z_{0\mathrm{h}}$ plane into 8 regions, the new scheme develops a group of equations with higher accuracy. The calculation error of $\zeta = f\left(Ri_{\mathrm{B}}, L_{0\mathrm{M}}, kB^{-1}\right)$ is always controlled to be within 5% (when $\zeta \le 0.5$) and 10% (when $\zeta > 0.5$). The calculation procedure is also simple, for a small Ri_{B} (i.e., $Ri_{\mathrm{B}} < Ri_{\mathrm{Bc1}}$), only one time computation of Eq. (23) and (24) will suffice. The maximum computation step is 6 times of Eq. (24) and one time of Eq. (23) when it is in region 1 or 7 and at the same time Ri_{B} is large (i.e., $Ri_{\mathrm{B}} > Ri_{\mathrm{Bc6}}$). Note that the Eq. (24) has only a maximum of 8 elements and a minimum of 4 elements so the calculation is still efficient. The new equations involve a large number of parameters which increase the complexity of coding. However, the effort of coding the new scheme is minimal as compared to its potential gain, which includes the accuracy of the new scheme and the avoidance of iterations. Besides, a compromise can be made between accuracy and complexity. For models that are not interested in high kB^{-1} values, region 1 and 2 (i.e., $10 \le z/z_0 \le 10^5$ and $-0.607 \le z_0/z_{0\mathrm{h}} \le 100$) have provided reasonable coverage (see Garratt, 1992; Launiainen, 1995), and the other 6 regions can be ignored. For example, in WRF model MM5 surface module, $z_{0\mathrm{h}} = z_0$ is assumed during the calculation of frictional velocity (Jiménez et al, 2012). While for models that include urban surface effects, it is better to keep all the regions. Further, CB05 probably is not the final solution for the surface flux calculation under stable stratification. The method used to derive non-iterative equations presented here can be used in future studies to transfer the new iterative algorithm to non-iterative equations.Overall, the new equations cover the full range of $-0.5 \le kB^{-1} \le 30$, $10 \le z/z_0 \le 10^5$ and stable condition (i.e., $0 < Ri_B \le 2.5$), and maintain high accuracy and efficiency. It is expected that its usage in climate and weather forecasting models can lead to better performance in surface flux calculation under stable conditions, especially over urban surfaces.

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Table 1. The 8 regions divided by z / z_0 and z_0 / z_{0h} values.

Region	z / z_0	z_0 / z_{0h}
1	10~160	0.607~100
2	160~10^5	0.607~100
3	10~80	100~10^7
4	80~10^5	100~10^7
5	10~40	10^7~10^{11}
6	40~10^5	10^7~10^{11}
7	10~40	10^{11}~1.07×10^{13}
8	40~10^5	10^{11}~1.07×10^{13}

Table 2. The coefficients of Eq. (24).

Region		C_{00}	C_{10}	C_{20}	C_{01}	C_{11}	C_{21}	C_{02}	C_{12}
1	Ri_{Bc1}	0.3095	-0.2852	0.07955	0.03388	-0.01605	0	0	-1.079E-4
	Ri_{Bc2}	0.3219	-0.2613	0.06753	0.04838	-0.03101	0.003908	-0.00178	0.001165
	Ri_{Bc3}	0.3545	-0.2569	0.06609	0.05837	-0.03934	0.005643	-0.003381	0.002194
	Ri_{Bc4}	0.439	-0.3133	0.08619	0.0893	-0.07112	0.01403	-0.005965	0.003806
	Ri_{Bc5}	0.6887	-0.5375	0.1616	0.1754	-0.1564	0.03489	-0.01277	0.008101
	Ri_{Bc6}	1.706	-1.62	0.5231	0.5124	-0.5026	0.1239	-0.03577	0.02238
2	Ri_{Bc1}	0	0.08606	-0.03048	0.09019	-0.07682	0.01693	0	0
	Ri_{Bc2}	0.2002	0	-0.01589	0	0.00367	0	0.005057	-0.002399
	Ri_{Bc3}	0.4499	0	-0.02397	0.0388	-0.01145	0	0	0
3	Ri_{Bc1}	0.3063	-0.2849	0.07886	0.03104	-0.01423	-5.632E-4	3.684E-6	-2.926E-6
	Ri_{Bc2}	0.3555	-0.3002	0.07855	0.02617	-0.004769	-0.004012	-1.298E-5	9.907E-6
	Ri_{Bc3}	0.5064	-0.4282	0.1229	0.02138	0	-0.00441	0	0
	Ri_{Bc4}	1.638	-1.743	0.5813	0.04471	-0.01874	0	0	0
4	Ri_{Bc1}	0.09742	0	-0.01096	0.04544	-0.03299	0.006383	0	0
	Ri_{Bc2}	0.1768	0	-0.01434	0.03558	-0.02059	0.003327	0	0
	Ri_{Bc3}	0.3636	0	-0.0224	0.04607	-0.02506	0.004152	0	0
5	Ri_{Bc1}	0	0	0	0.04825	-0.01677	-0.004762	-5.212E-4	2.768E-4
	Ri_{Bc2}	0	0	0.08807	0.05219	-0.01822	-0.01245	-8.5E-4	7.516E-4
	Ri_{Bc3}	0	0	0.1219	0.0583	-0.02373	-0.01224	-0.001081	9.539E-4
	Ri_{Bc4}	0	0	0.1609	0.07789	-0.04617	-0.00736	-0.001399	0.001238
	Ri_{Bc5}	0.4437	0	0	0.1349	-0.1388	0.03347	-0.00119	0.001095
6	Ri_{Bc1}	0	0	0	0.05594	-0.03245	0.005037	-3.654E-4	1.135E-4
	Ri_{Bc2}	0.1945	0	0	0.03347	-0.02116	0.002301	0	8.92E-5
	Ri_{Bc3}	0.4288	-0.1436	0.01635	0.03207	-0.01382	0.001571	1.326E-5	-6.424E-6
7	Ri_{Bc1}	0	0	0	0.03681	-0.007664	-0.005619	-1.211E-4	0
	Ri_{Bc2}	0	0	0	0.03655	0	-0.009977	-2.691E-4	1.057E-4
	Ri_{Bc3}	0	0	0	0.03822	0	-0.01036	-3.658E-4	1.769E-4
	Ri_{Bc4}	0	0	0	0.0384	0	-0.009243	-3.629E-4	1.471E-4
	Ri_{Bc5}	0	0	0	0.05616	-0.02275	0	-5.172E-4	2.261E-4
	Ri_{Bc6}	0	0	0	0.1472	-0.1144	0.02796	-0.001218	5.835E-4
8	Ri_{Bc1}	0	0	0	0.05139	-0.02991	0.004664	-2.135E-4	6.535E-5
	Ri_{Bc2}	0	0	0	0.04919	-0.0197	0.002011	-3.325E-4	7.974E-5
	Ri_{Bc3}	0.5775	-0.2236	0.03477	0.03805	-0.01617	0.00177	-2.191E-5	1.067E-5

Table 3. The coefficients of Eq. (23) for Region1.

	Region 1						
	Section 1	Section 2	Section 3	Section 4	Section 5	Section 6	Section 7
C_{000}	-1.134	0	0	0	0	0	0
C_{100}	31.1	86.35	-280.4	0	0	-17.32	-6.343
C_{200}	-71.16	0	3235	0	0	8.773	7.66
C_{300}	227.4	0	-6165	0	0	0	-0.7661
C_{001}	-0.2094	-11.53	-10.64	0	0	0	0.0125
C_{101}	3.293	194.9	193.8	0	1.113	0	-2.203
C_{201}	-20.11	-975.4	-1194	-12.37	-97.56	0	0.8896
C_{301}	14.42	1472	2161	0	159.4	0	-0.1273
C_{002}	0.1476	-2.535	-4.603	0	0	1.919	-0.00827
C_{102}	-0.07325	28.24	52.02	11.99	16.33	0	0.3327
C_{202}	0.5627	-61.13	-110.7	-15.63	-25.67	0.2679	-0.04613
C_{003}	-0.01178	-0.2378	-0.5367	-0.3157	-0.6447	-0.2892	0
C_{103}	0.0218	0.7405	1.503	0.2948	0.9718	0	-0.04968
C_{010}	1.405	13.6	30.26	0	6.821	10.27	7.513
C_{110}	-32.47	-316.2	-314.9	0	-57.13	0	0
C_{210}	46.59	1067	186	-108.1	227.3	0	-4.799
C_{310}	-38.25	-1494	0	317.8	-244	0	0.5598
C_{011}	-0.2286	8.023	9.038	0	0.9287	-3.457	-1.612
C_{111}	-1.097	-91.31	-87.06	-12.52	-17.88	-1.617	0
C_{211}	-0.3394	213.7	198.6	0	34.41	0	0
C_{012}	0	1.035	1.529	0	0.319	-0.07536	0.4666
C_{112}	0	-5.072	-7.439	-1.025	-2.452	0	0.0605
C_{013}	0	0.03622	0.07369	0.04669	0.08583	0.05146	-0.01808
C_{020}	0	-4.699	-10.71	-1.896	-2.195	-3.108	0
C_{120}	10.71	97.46	122.1	28.39	22.21	7.948	2.442
C_{220}	0	-152.4	-76.91	-14.19	-31.44	-2.985	0.1584
C_{021}	0	-1.704	-2.035	0	-0.1355	0.8751	0
C_{121}	0	9.069	8.248	2.214	1.976	0.3139	-0.04377
C_{022}	0	-0.09576	-0.1263	-0.01472	-0.04636	-0.05131	-0.0694
C_{030}	-0.007485	0.4446	1.015	0.3069	0.1708	0.2598	-0.1675
C_{130}	-0.9671	-7.991	-10.96	-3.635	-1.623	-0.8513	-0.2181
C_{031}	0.003402	0.1138	0.1426	-0.008769	0	-0.05427	0.05052

Table 4. Similar to Table 3, but for Region2.

	Region 2			
	Section 1	Section 2	Section 3	Section 4
C_{000}	0	0	0	0
C_{100}	0	0	41.53	0
C_{200}	0	0	0	0
C_{300}	0	0	0	0
C_{001}	0	0	-1.616	-2.57
C_{101}	0	-12.35	0	-2.91
C_{201}	0	0	0	0
C_{301}	0	0	0	0
C_{002}	0	0	0	0.874
C_{102}	0	0.5183	0	0.3377
C_{202}	0	0	0	0
C_{003}	0	0	0	-0.002092
C_{103}	0	0	0	-0.01343
C_{010}	0.9996	0.8247	0	7.453
C_{110}	0	0	15.82	5.4
C_{210}	56.57	112.5	-27.37	-1.623
C_{310}	0	0	0	0.1999
C_{011}	-0.1456	-0.09054	0	0
C_{111}	0	0	0	0.4753
C_{211}	-12.1	-2.249	0	0
C_{012}	0	0.01653	0	-0.2047
C_{112}	0.1303	0	0.02288	-0.02581
C_{013}	0	0	0	0
C_{020}	0	0	0.1062	-0.9043
C_{120}	0.295	0.8326	-0.9992	-0.3386
C_{220}	0	-9.554	1.56	0.04556
C_{021}	0.005508	0	0	0.04682
C_{121}	-0.0359	0.07022	0	-0.01924
C_{022}	4.067E-4	-0.001333	0	0.01217
C_{030}	0	0	0	0.03944
C_{130}	0	0	0	0.006516
C_{031}	0	0	0	-0.003571

Table 5. Similar to Table 3, but for Region 3.

	Region 3				
	Section 1	Section 2	Section 3	Section 4	Section 5
C_{000}	2.001	0	-68.85	-1.514	0
C_{100}	-0.7876	0	756.9	0	0
C_{200}	0	0	-1100	0	0
C_{300}	60.42	368.9	0	19.63	0
C_{001}	-0.1401	3.514	0	0.559	0
C_{101}	-0.1085	-8.524	-30.13	0	0
C_{201}	-2.065	-18.05	86.99	0	0
C_{301}	-2.98	-4.852	5.71	-2.424	0
C_{002}	0.01334	0.08174	0.7274	-0.002248	0
C_{102}	0.0213	0.5791	-2.554	0	0
C_{202}	0.1963	0.1207	-0.2169	0.1259	0
C_{003}	-3.704E-4	-0.007021	0.01587	8.267E-4	2.413E-4
C_{103}	-0.002957	0	0.003912	-0.004141	7.107E-5
C_{010}	-1.442	1.207	76.25	-8.751	0
C_{110}	1.047	-31.68	-874.1	51.96	1.905
C_{210}	0	32.78	1636	-76.51	-1.761
C_{310}	0	-25.65	-1040	27.69	0.3658
C_{011}	0	-2.096	4.942	-1.349	-0.05227
C_{111}	0	2.222	-17.32	1.297	0
C_{211}	-1.121	0.3871	14.97	-0.09621	0
C_{012}	0	-0.004486	-0.09096	0	0
C_{112}	0.0273	-0.06669	0.2281	0	0
C_{013}	0	0.001086	-0.002971	2.192E-4	0
C_{020}	0.6868	-0.07632	-21.66	3.734	2.165
C_{120}	0	14.32	232.4	-6.438	0.6139
C_{220}	3.82	2.353	-224.1	6.284	-0.1166
C_{021}	-0.01898	0.3396	-1.724	0.2422	-0.07307
C_{121}	-0.1228	-0.3281	3.144	-0.2272	0.005656
C_{022}	2.845E-4	-3.6E-4	-4.477E-4	0	0
C_{030}	-0.06543	0	1.875	-0.4111	-0.3134
C_{130}	0.1469	-1.505	-18.02	0.2556	0
C_{031}	0.00179	-0.01529	0.1523	-0.009961	0.008105

Table 6. Similar to Table 3, but for Region 4.

	Region 4			
	Section 1	Section 2	Section 3	Section 4
C_{000}	0	-3.528	0	0
C_{100}	0	0	0	0
C_{200}	0	0	0	-8.306
C_{300}	0	0	0	1.212
C_{001}	0	-0.2511	-1.018	0
C_{101}	0	0	0	0
C_{201}	-6.267	-10.06	0	0
C_{301}	0	0	0	0
C_{002}	0	0	0	0
C_{102}	0.09808	0.1809	0	0.0279
C_{202}	0	0	0	0
C_{003}	0	0	6.74E-5	6.853E-4
C_{103}	0	0	0.001341	-9.314E-4
C_{010}	0.5961	1.375	-2.404	5.253
C_{110}	0	2.951	41.12	7.626
C_{210}	18.49	68.09	-48.05	-0.2889
C_{310}	34.53	0	24.94	0.06073
C_{011}	-0.0845	0	-0.06671	-0.3959
C_{111}	-0.5106	-1.361	0	-0.07098
C_{211}	-0.3543	0	-0.1319	0.003821
C_{012}	0.004555	0.003711	0.006818	0
C_{112}	0	0	0	0
C_{013}	-9.402E-5	0	-1.788E-4	0
C_{020}	0.05628	-0.02359	0.5172	-0.5006
C_{120}	0.8075	0.305	-4.023	-0.7376
C_{220}	0	-3.765	2.074	0
C_{021}	0	-0.001535	0	0.04853
C_{121}	0.01631	0.07098	0	0.002956
C_{022}	-3.8E-5	-2.577E-4	0	0
C_{030}	-0.00189	0	-0.0192	0.01968
C_{130}	-0.03755	0	0.125	0.025
C_{031}	5.177E-5	0	0	-0.001897

Table 7. Similar to Table 3, but for Region 5.

	Region 5					
	Section 1	Section 2	Section 3	Section 4	Section 5	Section 6
C_{000}	0	0	-207.7	-587.1	0	0
C_{100}	0	77.11	880	2726	7.886	0
C_{200}	-2.541	-201.2	-1550	-3759	-0.5889	0
C_{300}	25.22	386.1	2201	1605	0	0
C_{001}	-0.03201	-0.6831	0	-9.376	-0.4057	0
C_{101}	0.1159	0	11.61	-4.513	0	0
C_{201}	-0.5745	-7.571	-96.51	70.55	-0.5218	0
C_{301}	-0.8502	-8.978	0	-58.16	0	0
C_{002}	0.00208	0.07136	0.5093	0.1711	0.01745	0
C_{102}	-0.001668	0	0.8873	-0.9373	-0.01349	0
C_{202}	0.03737	0.3442	0.2868	1.132	0.01468	0
C_{003}	-1.828E-5	0	-0.001909	-0.006865	0	0
C_{103}	-3.967E-4	-0.003421	-0.004313	-0.001126	0	0
C_{010}	0.4298	0	189.4	286.9	0	0
C_{110}	-0.03339	-31.72	-543.8	-903.7	0	0
C_{210}	0.05692	2.558	324	407.6	0	0
C_{310}	0	0	-80.25	260.2	0	0.08919
C_{011}	-0.0233	0	-5.403	0	0	0
C_{111}	0	2.695	14.95	14.82	0.2908	0
C_{211}	-0.3158	-2.449	-1.706	-26.07	0.1992	0
C_{012}	0	-0.05044	-0.4221	0.01062	-0.003177	0
C_{112}	0.007595	0.05465	0.164	0.2099	-0.00933	0
C_{013}	0	-6.869E-5	-0.00111	9.863E-4	0	0
C_{020}	0	0.3612	-53.83	-44.24	0.7321	2.053
C_{120}	0	0	89.42	98.98	2.304	0.2534
C_{220}	1.793	18.63	34.6	22.67	-2.456	-0.2585
C_{021}	0.00249	0.1236	2.704	-0.01096	-0.09448	-0.0338
C_{121}	-0.05666	-0.837	-4.573	-1.67	0.007636	0.004269
C_{022}	0	0.008316	0.0718	-0.01056	0.002124	0
C_{030}	0	-0.06987	4.95	2.138	0	-0.3116
C_{130}	0.129	0.8756	-3.112	-4.604	0	0.1241
C_{031}	0	-0.01959	-0.3287	0.054	0	0

Table 8. Similar to Table 3, but for Region 6.

	Region 6			
	Section 1	Section 2	Section 3	Section 4
C_{000}	0	0.4383	0	-6.744
C_{100}	-7.864	0	-41.74	8.8
C_{200}	0	0	177	-13.03
C_{300}	0	0	-118.2	2.203
C_{001}	-0.02699	0	0	-0.1139
C_{101}	0.7414	-4.81	-4.006	-0.06103
C_{201}	-1.114	5.094	-0.5102	0.2406
C_{301}	0	-1.159	0	-0.04635
C_{002}	0	0.04547	0	0.01341
C_{102}	0	0	0.0567	-0.002749
C_{202}	0	-0.1233	0.1868	5.316E-6
C_{003}	0	-5.595E-4	0.002457	-1.434E-4
C_{103}	1.281E-4	0.002459	-0.006455	0
C_{010}	0.244	0	0	6.511
C_{110}	1.743	0	27.45	6.369
C_{210}	4.749	44.44	-17.37	-0.175
C_{310}	11.28	0	-7.74	0.03419
C_{011}	0	0	0	-0.3147
C_{111}	-0.3093	0	0	-0.06781
C_{211}	-0.2208	-0.6068	0.0117	-2.026E-4
C_{012}	0	-0.005459	-0.01576	0.002444
C_{112}	0.003674	0	0.02102	2.616E-4
C_{013}	0	0	-1.975E-5	-5.149E-6
C_{020}	0.04168	0	-0.1563	-0.6219
C_{120}	0.4341	0.9983	-2.085	-0.598
C_{220}	0.6518	-2.874	0.3443	0.002868
C_{021}	-0.00208	-0.00152	0.03278	0.03359
C_{121}	0	0.01501	-0.0325	0.003178
C_{022}	2.895E-5	3.541E-4	5.167E-4	-1.423E-4
C_{030}	0	0.006587	0.008163	0.02407
C_{130}	-0.01307	-0.04253	0.0854	0.0188
C_{031}	1.425E-5	-3.659E-4	-0.001602	-0.001167

Table 9. Similar to Table 3, but for Region 7.

	Region 7						
	Section 1	Section 2	Section 3	Section 4	Section 5	Section 6	Section 7
C_{000}	-1.412	-4.502	-104.2	542.4	178.4	0	0
C_{100}	6.658	40.44	136.3	-1845	158.8	0	0
C_{200}	-5.68	37.42	233.3	2157	-480.9	0	0
C_{300}	11.9	0	0	0	0	0	0
C_{001}	0.1285	0.3067	13.8	-3.691	-31.49	0	0
C_{101}	-0.111	-5.444	-37.21	3.33	47.56	0	0
C_{201}	-0.2095	2.053	10.33	-45.62	-4.153	0	0
C_{301}	-0.3181	0	0	0	0	0	0
C_{002}	-0.004693	0.05302	-0.1157	0.1434	0.3998	0	0
C_{102}	0.004467	0	0.5542	0.4557	-0.8692	0	0
C_{202}	0.01324	-0.01586	-0.2568	0.08936	0.2504	0	0
C_{003}	6.64E-5	0	0	0	0	0	0
C_{103}	-2.023E-4	0	0	0	0	0	0
C_{010}	0.7122	1.663	16.56	-263.7	-37.94	0	0
C_{110}	-4.599	-28.1	0	677.7	-147.8	20.56	0
C_{210}	2.705	-11.02	-114.4	-644.2	144.7	-13.42	0
C_{310}	0	0	0	0	0	3.002	0
C_{011}	-0.04962	0.1172	-3.238	4.44	9.904	-0.5254	0.06758
C_{111}	0.01147	1.979	7.578	-3.037	-7.914	0	0
C_{211}	-0.1621	-0.7285	0	10.93	-2.224	0	0.003671
C_{012}	0.001459	-0.0293	0	-0.08875	-0.1235	0	0
C_{112}	0.003514	0.01334	-0.06568	-0.1436	0.1631	0	-6.967E-4
C_{013}	-2.01E-5	0	0	0	0	2.282E-4	0
C_{020}	0.003692	-0.4475	0	32.93	0	0	0
C_{120}	1.299	5.193	-0.1495	-56.58	23.51	-2.349	0.6983
C_{220}	0.6516	5.593	18.12	53.14	-2.645	0.628	-0.1455
C_{021}	0	-0.009728	0.167	-0.951	-0.7278	0.2176	0
C_{121}	-0.03414	-0.3375	-0.6387	0	-0.1801	0.02067	0
C_{022}	2.84E-5	0.00347	0.00428	0.02119	0.008599	-0.005396	-4.282E-4
C_{030}	6.293E-4	0	0	0	0	-0.4148	0
C_{130}	-0.02559	0	0	0	0	0.02245	0
C_{031}	0	0	0	0	0	0.01163	0

Table 10. Similar to Table 3, but for Region 8.

	Region 8			
	Section 1	Section 2	Section 3	Section 4
C_{000}	-3.13	-49.55	0	0
C_{100}	5.26	97.14	0	0
C_{200}	-29.85	352.5	10.72	0
C_{300}	57.04	-573.4	0	0
C_{001}	0.2176	2.052	0	0
C_{101}	-0.00898	-21.41	0	0
C_{201}	-1.756	13.12	0	0
C_{301}	-1.663	20.82	-1.354	0
C_{002}	-0.007271	0.1357	-0.06227	0
C_{102}	0.0304	0.238	0	-0.01477
C_{202}	0.05349	-0.7316	0.08799	-0.001292
C_{003}	8.978E-5	-0.003367	0.002359	0
C_{103}	-6.252E-4	0.006023	-0.002387	3.921E-4
C_{010}	0.9846	14.57	-0.2492	0
C_{110}	-1.011	0	19.79	0
C_{210}	14.45	0	-18.86	-0.8522
C_{310}	4.433	-54.39	9.463	0.1065
C_{011}	-0.05083	-0.8911	0	0
C_{111}	-0.2604	1.478	0	0.374
C_{211}	-0.2977	2.13	-0.3291	0.004036
C_{012}	0.001361	-9.36E-4	0	0.002528
C_{112}	0.00375	-0.04272	0.01369	-0.006853
C_{013}	-1.464E-5	1.939E-4	-2.41E-4	-8.747E-5
C_{020}	-0.004659	-1.165	0	0
C_{120}	0.6393	0	-1.689	-0.4307
C_{220}	0	-3.616	1.036	0.01469
C_{021}	0	0.06747	0.00194	0.001642
C_{121}	0	0.01581	-0.02897	0
C_{022}	0	-3.126E-4	8.316E-4	0
C_{030}	8.014E-4	0.03485	0.01694	0
C_{130}	-0.01934	0	0.06734	0.01348
C_{031}	0	-0.001713	-0.001447	0

Table 11. Summarization of the characteristics of the four methods. Calculation time is the time each method needs for computing ζ from Ri_B, z_0 and z_{0h} in the range $0 < Ri_B \le 2.5$, $10 \le z/z_0 \le 10^5$ and $-0.5 \le \log(z_0/z_{0h}) \le 30$ with the interval of 0.01 for Ri_B, 0.035 for $\log(z/z_0)$ and 0.1 for $\log(z_0/z_{0h})$. The calculation is performed on a desktop computer with an Intel Core i5 processor, and note that the calculation time can vary with different computer.

Method	Calculation time	Maximum $\Delta\zeta$	Average $\Delta\zeta$	Characteristics and suggestion
CB05 with ultimate iteration	6260 s	N/A	N/A	Current optimal method, but with high computational cost. Use this method when computing power is not an issue.
CB05 with 5 steps iteration	3960 s	exceeds 50%	exceeds 15%	Lower computational cost, but add more uncertainty in the calculation procedure of CB05.
WRL12	261 s	exceeds 50%	exceeds 15%	Much lower computational cost, but add more uncertainty in the calculation procedure of CB05.
New equations	549 s	smaller than 5% (when $\zeta \le 0.5$) and 10% (when $\zeta > 0.5$)	smaller than 2%	Low computational cost, error in the calculation procedure of CB05 is controlled within 10%. Use this method to have an optimal compromise between accuracy and computational cost.

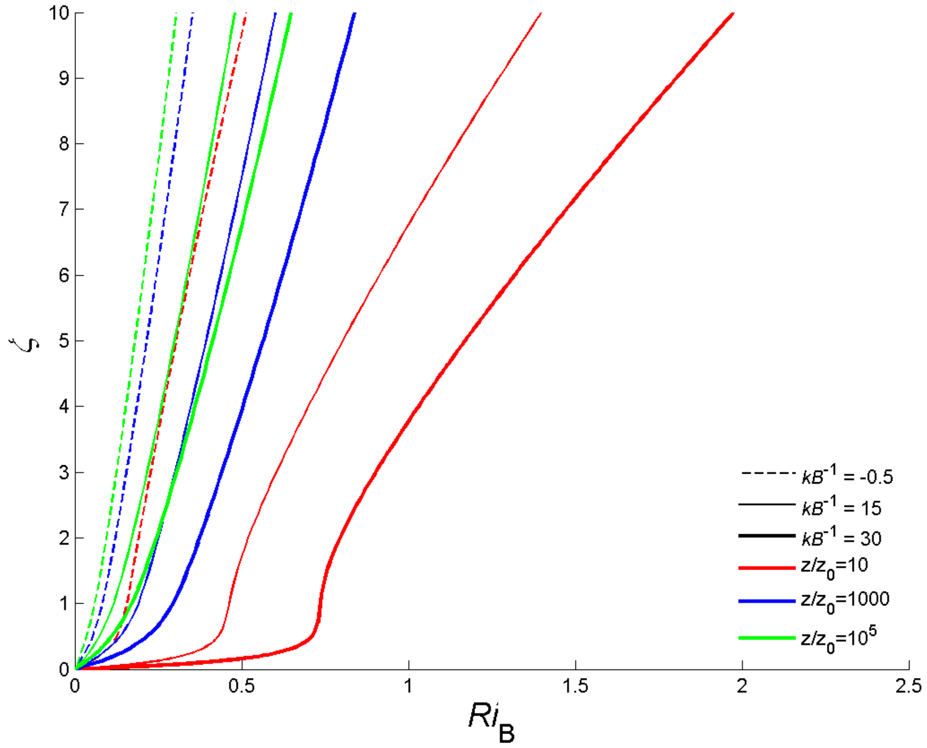

Figure 1. The relationship between Ri_B and ζ from the precise results of CB05

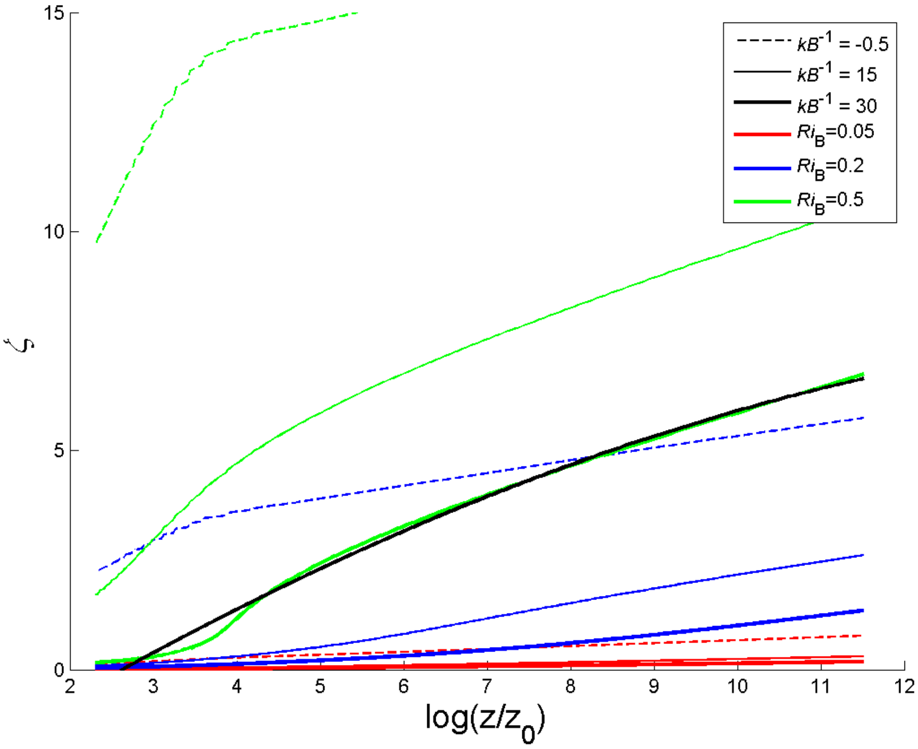

Figure 2. The relationship between log(z/z_0) and ζ from the precise results of CB05. Black line indicates the cubic fit of curve kB^{-1}=30 and Ri_B=0.5 with least square method.

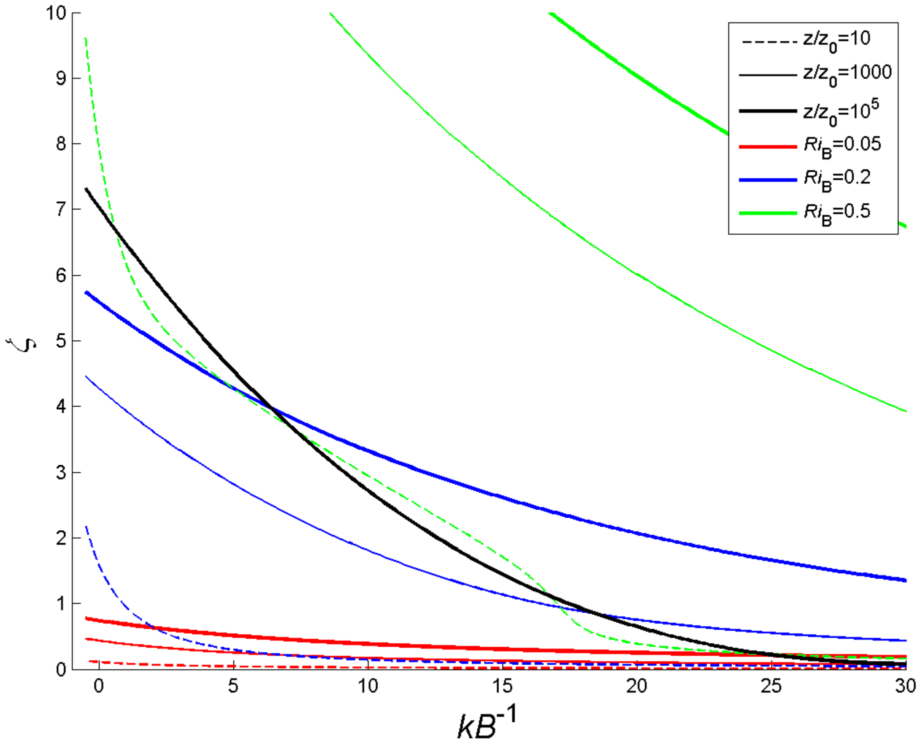

Figure 3. The relationship between log(z/z_{0h}) (i.e., kB^{-1}) and ζ from the precise results of CB05. Black line indicates the cubic fit of curve z/z_0=10 and Ri_B=0.5 with least square method.

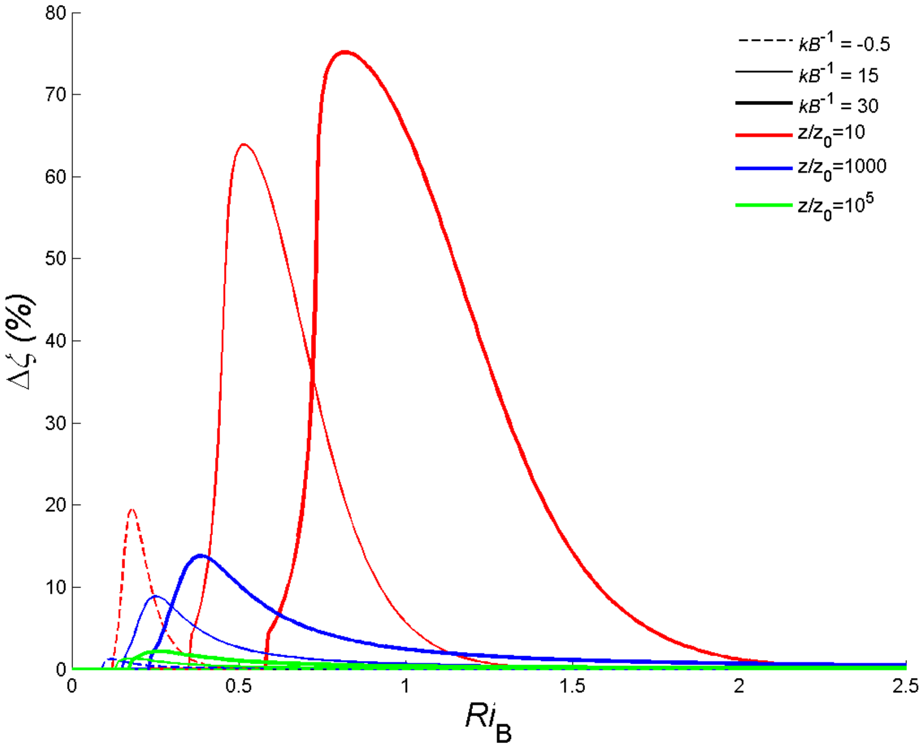

Figure 4. Relative error after 5 steps of iteration with CB05 equations under certain z_0 and z_{0h} conditions.

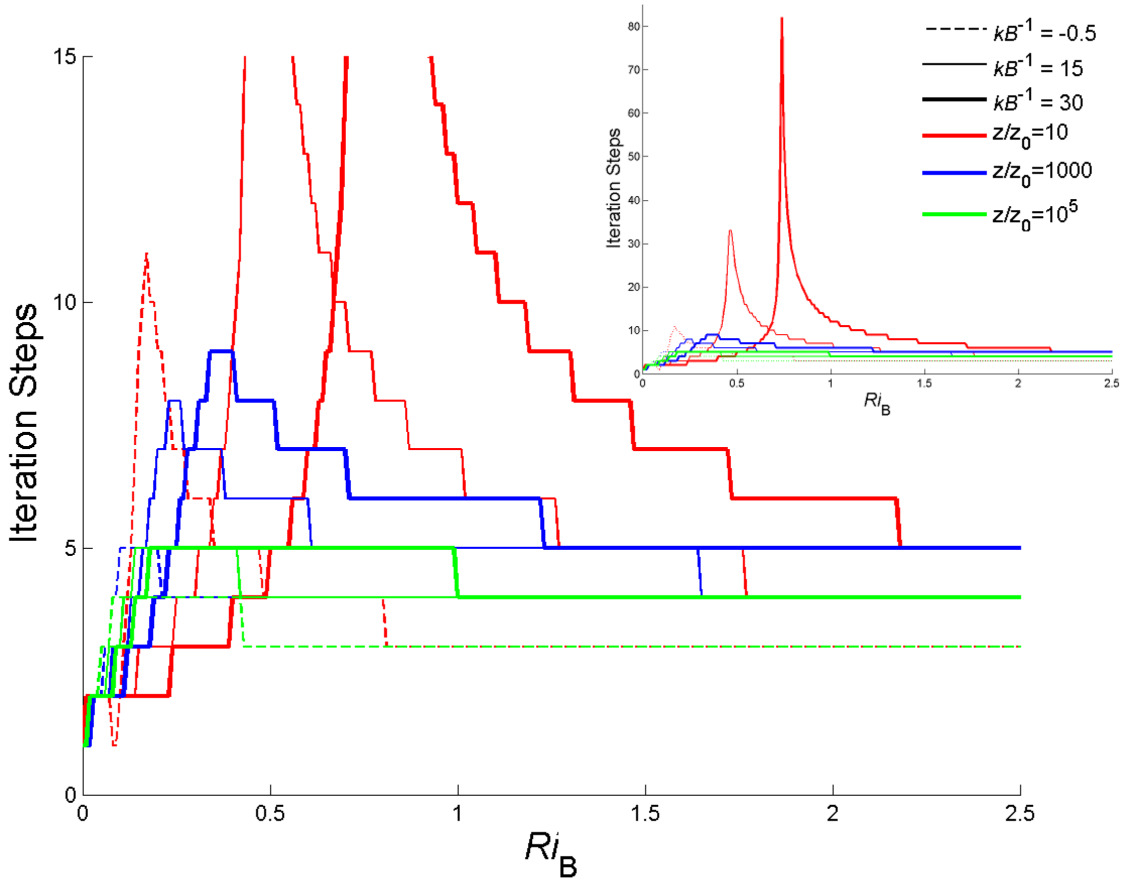

Figure 5. Steps needed to converge into 5% relative error with CB05 equations under certain z_0 and z_{0h} conditions. The inset shows the whole perspective.

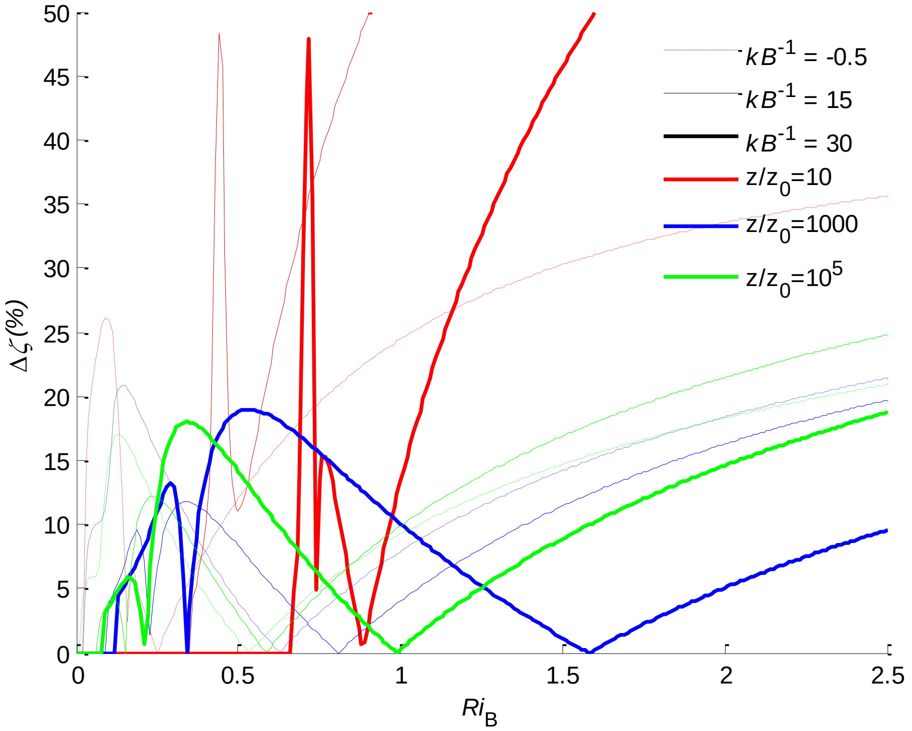

Figure 6. Relative error with WRL12 equations.

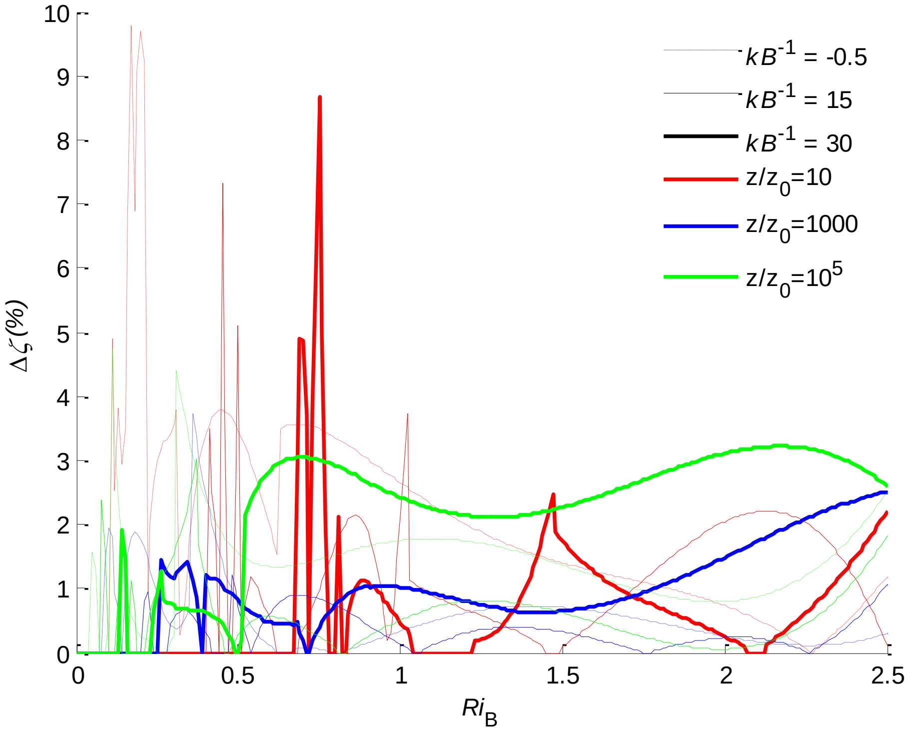

Figure 7. Relative error with new equations.

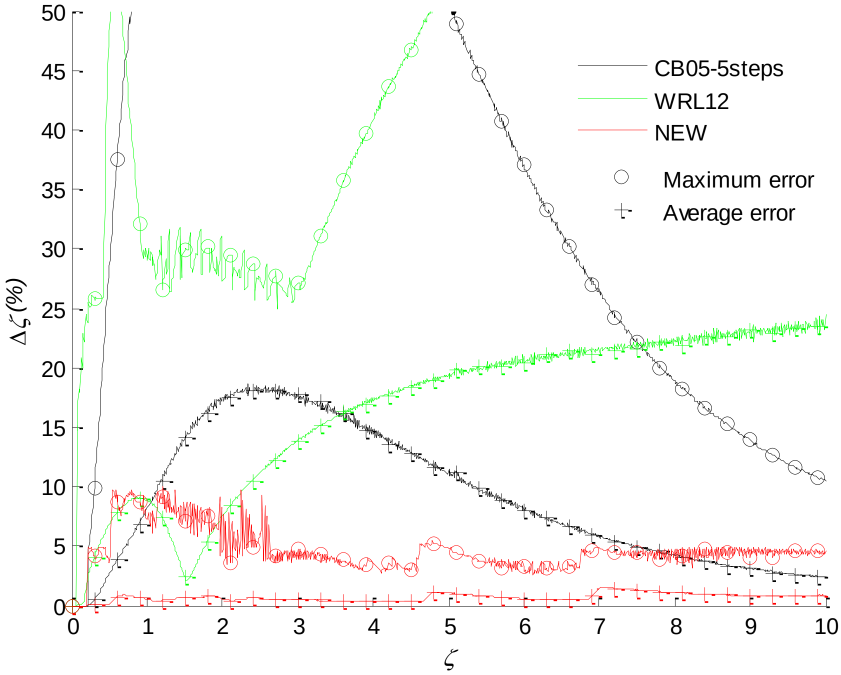

Figure 8. Maximum (circles) and average (crosses) relative error of ζ for CB05 with 5 steps iteration (black lines), WRL12 (green lines) and the new scheme (red lines). Errors larger than 50% are not shown.

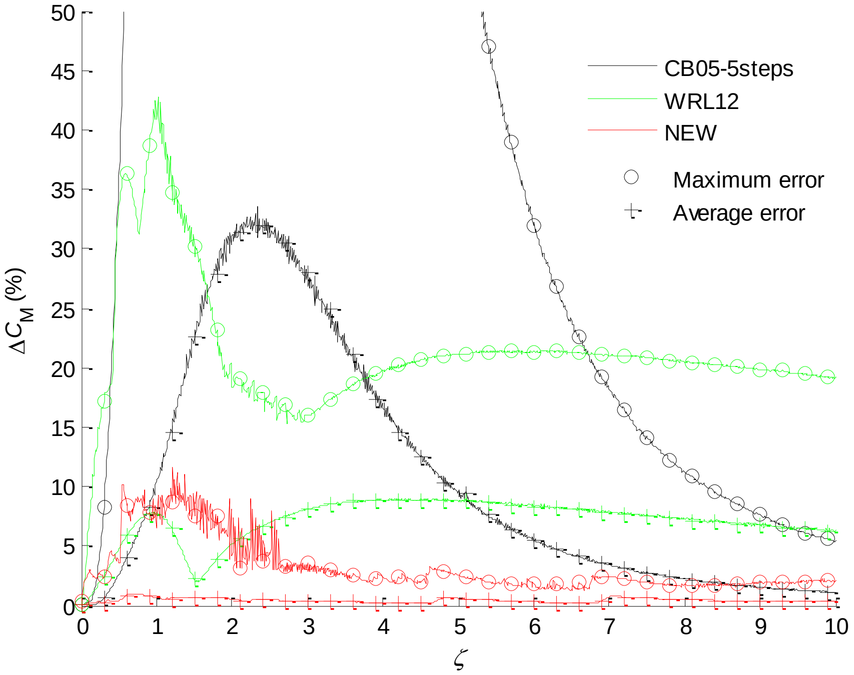

Figure 9.Similar toFigure 8 but for C_M.

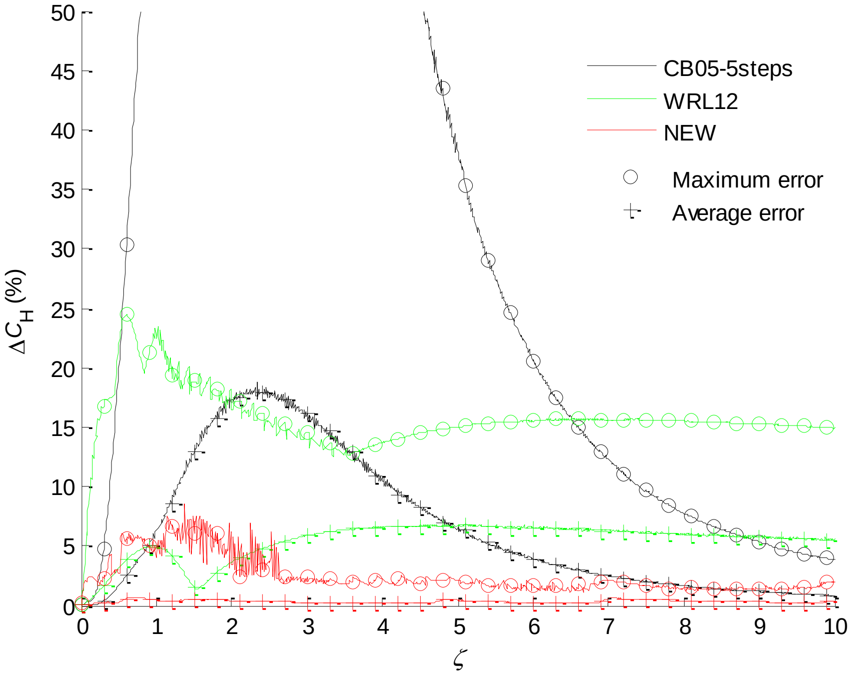

Figure 10.Similar toFigure 8 but for C_H.