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Supplement of

Parameterizing microphysical effects on variances and covariances of moisture and heat content using a multivariate probability density function: a study with CLUBB (tag MVCS)

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PDF Forms and Solved Integral Equations

This Supplement provides the functional forms of the PDFs (Section S1 through Section S4) and the solved integral equations (Section S5 through Section S8) that are necessary to solve the microphysical covariance integrals listed in Appendix A of the associated article. These equations need to be substituted into the appropriate places in the integrals in Appendix A. The text in Appendix A references the appropriate equations in this Supplement. The equations are presented in this manner because providing complete formulas in the associated article would have led to equations that are too lengthy and have too many sub-forms. Additionally, this style of presentation mirrors the structure of the source code implementation, providing clearer mathematical documentation of the source code. Interested readers may view the source code for more details on the implementation of these integrals.

S1 Functional Form of a Quadrivariate PDF

There is one type of quadrivariate PDF used in the equation set. It is a quadrivariate normal-normal-lognormal-lognormal distribution, meaning that the individual marginal of x_1 is a normal distribution, the individual marginal of x_2 is a normal distribution, the individual marginal of x_3 is a lognormal distribution, and the individual marginal of x_4 is a lognormal distribution. The functional form of this type of PDF is given by:

$$P_{NNLL(i)}\left(x_1, x_2, x_3, x_4\right) = \frac{\exp\left\{-\frac{1}{2}\lambda_{NNLL}\right\}}{\left(2\pi\right)^2 \sigma_{x_1(i)}\sigma_{x_2(i)}\tilde{\sigma}_{x_3(i)}\tilde{\sigma}_{x_4(i)}C_{Q1}x_3x_4}; \tag{S1}$$

where:

$$\begin{aligned}
\lambda_{NNLL} = \frac{1}{C_{Q1}^2}\Big[& C_{Q2}\left(x_1 - \mu_{x_1(i)}\right)^2 + C_{Q3}\left(x_2 - \mu_{x_2(i)}\right)^2 + C_{Q4}\left(\ln x_3 - \tilde{\mu}_{x_3(i)}\right)^2 \\
& + C_{Q5}\left(\ln x_4 - \tilde{\mu}_{x_4(i)}\right)^2 + C_{Q6}\left(x_1 - \mu_{x_1(i)}\right)\left(x_2 - \mu_{x_2(i)}\right) \\
& + C_{Q7}\left(x_1 - \mu_{x_1(i)}\right)\left(\ln x_3 - \tilde{\mu}_{x_3(i)}\right) + C_{Q8}\left(x_1 - \mu_{x_1(i)}\right)\left(\ln x_4 - \tilde{\mu}_{x_4(i)}\right) \\
& + C_{Q9}\left(x_2 - \mu_{x_2(i)}\right)\left(\ln x_3 - \tilde{\mu}_{x_3(i)}\right) + C_{Q10}\left(x_2 - \mu_{x_2(i)}\right)\left(\ln x_4 - \tilde{\mu}_{x_4(i)}\right) \\
& + C_{Q11}\left(\ln x_3 - \tilde{\mu}_{x_3(i)}\right)\left(\ln x_4 - \tilde{\mu}_{x_4(i)}\right)\Big];
\end{aligned}$$

and where:

$$\begin{aligned}
C_{Q1} = \Big[& 1 - \left(\rho_{x_1,x_2(i)}^2 + \tilde{\rho}_{x_1,x_3(i)}^2 + \tilde{\rho}_{x_1,x_4(i)}^2 + \tilde{\rho}_{x_2,x_3(i)}^2 + \tilde{\rho}_{x_2,x_4(i)}^2 + \tilde{\rho}_{x_3,x_4(i)}^2\right) \\
& + 2\rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_3(i)} + 2\rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_2,x_4(i)} \\
& + 2\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_3,x_4(i)} + 2\tilde{\rho}_{x_2,x_3(i)}\tilde{\rho}_{x_2,x_4(i)}\tilde{\rho}_{x_3,x_4(i)} + \rho_{x_1,x_2(i)}^2\tilde{\rho}_{x_3,x_4(i)}^2 \\
& + \tilde{\rho}_{x_1,x_3(i)}^2\tilde{\rho}_{x_2,x_4(i)}^2 + \tilde{\rho}_{x_1,x_4(i)}^2\tilde{\rho}_{x_2,x_3(i)}^2 - 2\rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_4(i)}\tilde{\rho}_{x_3,x_4(i)} \\
& - 2\rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_2,x_3(i)}\tilde{\rho}_{x_3,x_4(i)} - 2\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_2,x_3(i)}\tilde{\rho}_{x_2,x_4(i)}\Big]^{\frac{1}{2}};
\end{aligned}$$

$$C_{Q2} = \frac{1}{\sigma_{x_1(i)}^2}\Big[1 - \left(\tilde{\rho}_{x_2,x_3(i)}^2 + \tilde{\rho}_{x_2,x_4(i)}^2 + \tilde{\rho}_{x_3,x_4(i)}^2\right) + 2\tilde{\rho}_{x_2,x_3(i)}\tilde{\rho}_{x_2,x_4(i)}\tilde{\rho}_{x_3,x_4(i)}\Big];$$

$$C_{Q3} = \frac{1}{\sigma^2_{x_2(i)}}\Big[1 - \big(\tilde{\rho}^2_{x_1,x_3(i)} + \tilde{\rho}^2_{x_1,x_4(i)} + \tilde{\rho}^2_{x_3,x_4(i)}\big) + 2\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_3,x_4(i)}\Big];$$

$$C_{Q4} = \frac{1}{\tilde{\sigma}^2_{x_3(i)}}\Big[1 - \big(\rho^2_{x_1,x_2(i)} + \tilde{\rho}^2_{x_1,x_4(i)} + \tilde{\rho}^2_{x_2,x_4(i)}\big) + 2\rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_2,x_4(i)}\Big];$$

$$C_{Q5} = \frac{1}{\tilde{\sigma}^2_{x_4(i)}}\Big[1 - \big(\rho^2_{x_1,x_2(i)} + \tilde{\rho}^2_{x_1,x_3(i)} + \tilde{\rho}^2_{x_2,x_3(i)}\big) + 2\rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_3(i)}\Big];$$

$$\begin{aligned} C_{Q6} = \frac{2}{\sigma_{x_1(i)}\sigma_{x_2(i)}}\Big(&\rho_{x_1,x_2(i)}\tilde{\rho}^2_{x_3,x_4(i)} - \tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_2,x_3(i)}\tilde{\rho}_{x_3,x_4(i)} \\ &- \tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_4(i)}\tilde{\rho}_{x_3,x_4(i)} + \tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_3(i)} + \tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_2,x_4(i)} \\ &- \rho_{x_1,x_2(i)}\Big); \end{aligned}$$

$$\begin{aligned} C_{Q7} = \frac{2}{\sigma_{x_1(i)}\tilde{\sigma}_{x_3(i)}}\Big(&\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}^2_{x_2,x_4(i)} - \rho_{x_1,x_2(i)}\tilde{\rho}_{x_2,x_4(i)}\tilde{\rho}_{x_3,x_4(i)} \\ &- \tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_2,x_3(i)}\tilde{\rho}_{x_2,x_4(i)} + \rho_{x_1,x_2(i)}\tilde{\rho}_{x_2,x_3(i)} + \tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_3,x_4(i)} \\ &- \tilde{\rho}_{x_1,x_3(i)}\Big); \end{aligned}$$

$$\begin{aligned} C_{Q8} = \frac{2}{\sigma_{x_1(i)}\tilde{\sigma}_{x_4(i)}}\Big(&\tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}^2_{x_2,x_3(i)} - \rho_{x_1,x_2(i)}\tilde{\rho}_{x_2,x_3(i)}\tilde{\rho}_{x_3,x_4(i)} \\ &- \tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_3(i)}\tilde{\rho}_{x_2,x_4(i)} + \rho_{x_1,x_2(i)}\tilde{\rho}_{x_2,x_4(i)} + \tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_3,x_4(i)} \\ &- \tilde{\rho}_{x_1,x_4(i)}\Big); \end{aligned}$$

$$\begin{aligned} C_{Q9} = \frac{2}{\sigma_{x_2(i)}\tilde{\sigma}_{x_3(i)}}\Big(&\tilde{\rho}^2_{x_1,x_4(i)}\tilde{\rho}_{x_2,x_3(i)} - \rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_3,x_4(i)} \\ &- \tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_2,x_4(i)} + \tilde{\rho}_{x_2,x_4(i)}\tilde{\rho}_{x_3,x_4(i)} + \rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_3(i)} \\ &- \tilde{\rho}_{x_2,x_3(i)}\Big); \end{aligned}$$

$$\begin{aligned} C_{Q10} = \frac{2}{\sigma_{x_2(i)}\tilde{\sigma}_{x_4(i)}}\Big(&\tilde{\rho}^2_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_4(i)} - \rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_3,x_4(i)} \\ &- \tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_2,x_3(i)} + \tilde{\rho}_{x_2,x_3(i)}\tilde{\rho}_{x_3,x_4(i)} + \rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_4(i)} \\ &- \tilde{\rho}_{x_2,x_4(i)}\Big); \end{aligned}$$

$$C_{Q11} = \frac{2}{\tilde{\sigma}_{x_3(i)}\tilde{\sigma}_{x_4(i)}}\Big(\rho^2_{x_1,x_2(i)}\tilde{\rho}_{x_3,x_4(i)} - \rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_4(i)}\tilde{\rho}_{x_2,x_3(i)}$$
$$- \rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_4(i)} + \tilde{\rho}_{x_2,x_3(i)}\tilde{\rho}_{x_2,x_4(i)} + \tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_1,x_4(i)}$$
$$- \tilde{\rho}_{x_3,x_4(i)}\Big).$$

In Eq. (S1), $\mu_{x_1(i)}$ is the mean of x_1 in the ith component, $\mu_{x_2(i)}$ is the mean of x_2 in the ith component, $\tilde{\mu}_{x_3(i)}$ is the mean of $\ln x_3$ in the ith component, and $\tilde{\mu}_{x_4(i)}$ is the mean of $\ln x_4$ in the ith component. The ith component standard deviation of x_1 is $\sigma_{x_1(i)}$, the ith component standard deviation of x_2 is $\sigma_{x_2(i)}$, the ith component standard deviation of $\ln x_3$ is $\tilde{\sigma}_{x_3(i)}$, and the ith component standard deviation of $\ln x_4$ is $\tilde{\sigma}_{x_4(i)}$. The ith component correlation of x_1 and x_2 is $\rho_{x_1,x_2(i)}$, the ith component correlation of x_1 and $\ln x_3$ is $\tilde{\rho}_{x_1,x_3(i)}$, the ith component correlation of x_1 and $\ln x_4$ is $\tilde{\rho}_{x_1,x_4(i)}$, the ith component correlation of x_2 and $\ln x_3$ is $\tilde{\rho}_{x_2,x_3(i)}$, the ith component correlation of x_2 and $\ln x_4$ is $\tilde{\rho}_{x_2,x_4(i)}$, and the ith component correlation of $\ln x_3$ and $\ln x_4$ is $\tilde{\rho}_{x_3,x_4(i)}$.

S2 Functional Form of Trivariate PDFs

There are two types of trivariate PDFs used in the equation set. The first one is a trivariate normal-normal-lognormal distribution, meaning that the individual marginal of x_1 is a normal distribution, the individual marginal of x_2 is a normal distribution, and the individual marginal of x_3 is a lognormal distribution. The functional form of this type of PDF is given by:

$$P_{NNL(i)}\left(x_1, x_2, x_3\right) = \frac{\exp\left\{-\frac{1}{2}\lambda_{NNL}\right\}}{(2\pi)^{\frac{3}{2}}\,\sigma_{x_1(i)}\sigma_{x_2(i)}\tilde{\sigma}_{x_3(i)}C_{T1}x_3}; \tag{S2}$$

where:

$$\lambda_{NNL} = \frac{1}{C_{T1}^2}\Big[C_{T2}\left(x_1-\mu_{x1(i)}\right)^2 + C_{T3}\left(x_2-\mu_{x2(i)}\right)^2 + C_{T4}\left(\ln x_3-\tilde{\mu}_{x3(i)}\right)^2$$
$$+ C_{T5}\left(x_1-\mu_{x1(i)}\right)\left(x_2-\mu_{x2(i)}\right) + C_{T6}\left(x_1-\mu_{x1(i)}\right)\left(\ln x_3-\tilde{\mu}_{x3(i)}\right)$$
$$+ C_{T7}\left(x_2-\mu_{x2(i)}\right)\left(\ln x_3-\tilde{\mu}_{x3(i)}\right)\Big];$$

and where:

$$C_{T1} = \left[1-\left(\rho_{x_1,x_2(i)}^2+\tilde{\rho}_{x_1,x_3(i)}^2+\tilde{\rho}_{x_2,x_3(i)}^2\right)+2\rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_3(i)}\right]^{\frac{1}{2}};$$

$$C_{T2} = \frac{1-\tilde{\rho}_{x_2,x_3(i)}^2}{\sigma_{x1(i)}^2}; \qquad C_{T3} = \frac{1-\tilde{\rho}_{x_1,x_3(i)}^2}{\sigma_{x2(i)}^2}; \qquad C_{T4} = \frac{1-\rho_{x_1,x_2(i)}^2}{\tilde{\sigma}_{x3(i)}^2};$$

$$C_{T5} = \frac{2\left(\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_3(i)}-\rho_{x_1,x_2(i)}\right)}{\sigma_{x_{1(i)}}\sigma_{x_{2(i)}}}; \quad C_{T6} = \frac{2\left(\rho_{x_1,x_2(i)}\tilde{\rho}_{x_2,x_3(i)}-\tilde{\rho}_{x_1,x_3(i)}\right)}{\sigma_{x_{1(i)}}\tilde{\sigma}_{x_{3(i)}}};$$

and $$C_{T7} = \frac{2\left(\rho_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_3(i)}-\tilde{\rho}_{x_2,x_3(i)}\right)}{\sigma_{x_{2(i)}}\tilde{\sigma}_{x_{3(i)}}}.$$

In Eq. (S2), $\mu_{x_1(i)}$ is the mean of x_1 in the ith component, $\mu_{x_2(i)}$ is the mean of x_2 in the ith component, and $\tilde{\mu}_{x_3(i)}$ is the mean of $\ln x_3$ in the ith component. The ith component standard deviation of x_1 is $\sigma_{x_1(i)}$, the ith component standard deviation of x_2 is $\sigma_{x_2(i)}$, and the ith component standard deviation of $\ln x_3$ is $\tilde{\sigma}_{x_3(i)}$. The ith component correlation of x_1 and x_2 is $\rho_{x_1,x_2(i)}$, the ith component correlation of x_1 and $\ln x_3$ is $\tilde{\rho}_{x_1,x_3(i)}$, and the ith component correlation of x_2 and $\ln x_3$ is $\tilde{\rho}_{x_2,x_3(i)}$.

The second type of trivariate PDF used in the equation set is a trivariate normal-lognormal-lognormal distribution, meaning that the individual marginal of x_1 is a normal distribution, the individual marginal of x_2 is a lognormal distribution, and the individual marginal of x_3 is a lognormal distribution. The functional form of this type of PDF is given by:

$$P_{NLL(i)}\left(x_1,x_2,x_3\right)=\frac{\exp\left\{-\frac{1}{2}\lambda_{NLL}\right\}}{(2\pi)^{\frac{3}{2}}\,\sigma_{x_1(i)}\tilde{\sigma}_{x_2(i)}\tilde{\sigma}_{x_3(i)}C_{t1}x_2x_3}; \tag{S3}$$

where:

$$\begin{aligned}\lambda_{NLL}=\frac{1}{C_{t1}^2}\Big[&C_{t2}\left(x_1-\mu_{x_1(i)}\right)^2+C_{t3}\left(\ln x_2-\tilde{\mu}_{x_2(i)}\right)^2+C_{t4}\left(\ln x_3-\tilde{\mu}_{x_3(i)}\right)^2\\&+C_{t5}\left(x_1-\mu_{x_1(i)}\right)\left(\ln x_2-\tilde{\mu}_{x_2(i)}\right)+C_{t6}\left(x_1-\mu_{x_1(i)}\right)\left(\ln x_3-\tilde{\mu}_{x_3(i)}\right)\\&+C_{t7}\left(\ln x_2-\tilde{\mu}_{x_2(i)}\right)\left(\ln x_3-\tilde{\mu}_{x_3(i)}\right)\Big];\end{aligned}$$

and where:

$$C_{t1}=\left[1-\left(\tilde{\rho}_{x_1,x_2(i)}^2+\tilde{\rho}_{x_1,x_3(i)}^2+\tilde{\rho}_{x_2,x_3(i)}^2\right)+2\tilde{\rho}_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_3(i)}\right]^{\frac{1}{2}};$$

$$C_{t2}=\frac{1-\tilde{\rho}_{x_2,x_3(i)}^2}{\sigma_{x_1(i)}^2};\qquad C_{t3}=\frac{1-\tilde{\rho}_{x_1,x_3(i)}^2}{\tilde{\sigma}_{x_2(i)}^2};\qquad C_{t4}=\frac{1-\tilde{\rho}_{x_1,x_2(i)}^2}{\tilde{\sigma}_{x_3(i)}^2};$$

$$C_{t5}=\frac{2\left(\tilde{\rho}_{x_1,x_3(i)}\tilde{\rho}_{x_2,x_3(i)}-\tilde{\rho}_{x_1,x_2(i)}\right)}{\sigma_{x_{1(i)}}\tilde{\sigma}_{x_{2(i)}}};\quad C_{t6}=\frac{2\left(\tilde{\rho}_{x_1,x_2(i)}\tilde{\rho}_{x_2,x_3(i)}-\tilde{\rho}_{x_1,x_3(i)}\right)}{\sigma_{x_{1(i)}}\tilde{\sigma}_{x_{3(i)}}};$$

$$\text{and}\qquad C_{t7}=\frac{2\left(\tilde{\rho}_{x_1,x_2(i)}\tilde{\rho}_{x_1,x_3(i)}-\tilde{\rho}_{x_2,x_3(i)}\right)}{\tilde{\sigma}_{x_{2(i)}}\tilde{\sigma}_{x_{3(i)}}}.$$

In Eq. (S2), $\mu_{x_1(i)}$ is the mean of x_1 in the ith component, $\tilde{\mu}_{x_2(i)}$ is the mean of $\ln x_2$ in the ith component, and $\tilde{\mu}_{x_3(i)}$ is the mean of $\ln x_3$ in the ith component. The ith component standard deviation of x_1 is $\sigma_{x_1(i)}$, the ith component standard deviation of $\ln x_2$ is $\tilde{\sigma}_{x_2(i)}$, and the ith component standard deviation of $\ln x_3$ is $\tilde{\sigma}_{x_3(i)}$. The ith component correlation of x_1 and $\ln x_2$ is $\tilde{\rho}_{x_1,x_2(i)}$, the ith component correlation of x_1 and $\ln x_3$ is $\tilde{\rho}_{x_1,x_3(i)}$, and the ith component correlation of $\ln x_2$ and $\ln x_3$ is $\tilde{\rho}_{x_2,x_3(i)}$.

S3 Functional Form of Bivariate PDFs

There are three types of bivariate PDFs used in the equation set. The first one is a bivariate normal distribution, meaning that the individual marginal for each of x_1 and x_2 is a normal distribution. The functional form of this type of PDF is given by:

$$P_{NN(i)}\left(x_1, x_2\right) = \frac{\exp\left\{-\frac{1}{2}\lambda_{NN}\right\}}{2\pi\sigma_{x_1(i)}\sigma_{x_2(i)}\left(1-\rho^2_{x_1,x_2(i)}\right)^{\frac{1}{2}}}; \quad \text{where} \tag{S4}$$

$$\begin{aligned}\lambda_{NN} = \frac{1}{1-\rho^2_{x_1,x_2(i)}}\Bigg[&\frac{1}{\sigma^2_{x_1(i)}}\left(x_1-\mu_{x_1(i)}\right)^2 + \frac{1}{\sigma^2_{x_2(i)}}\left(x_2-\mu_{x_2(i)}\right)^2 \\ &- \frac{2\rho_{x_1,x_2(i)}}{\sigma_{x_1(i)}\sigma_{x_2(i)}}\left(x_1-\mu_{x_1(i)}\right)\left(x_2-\mu_{x_2(i)}\right)\Bigg];\end{aligned}$$

where the ith component mean of x_1 is $\mu_{x_1(i)}$, the ith component mean of x_2 is $\mu_{x_2(i)}$, the ith component standard deviation of x_1 is $\sigma_{x_1(i)}$, the ith component standard deviation of x_2 is $\sigma_{x_2(i)}$, and the ith component correlation of x_1 and x_2 is $\rho_{x_1,x_2(i)}$.

The second type of bivariate PDF used in the equation set is a bivariate normal-lognormal distribution, meaning that the individual marginal of x_1 is a normal distribution and the individual marginal of x_2 is a lognormal distribution. The functional form of this type of PDF is given by:

$$P_{NL(i)}\left(x_1, x_2\right) = \frac{\exp\left\{-\frac{1}{2}\lambda_{NL}\right\}}{2\pi\sigma_{x_1(i)}\tilde{\sigma}_{x_2(i)}\left(1-\tilde{\rho}^2_{x_1,x_2(i)}\right)^{\frac{1}{2}}x_2}; \quad \text{where} \tag{S5}$$

$$\begin{aligned}\lambda_{NL} = \frac{1}{1-\tilde{\rho}^2_{x_1,x_2(i)}}\Bigg[&\frac{1}{\sigma^2_{x_1(i)}}\left(x_1-\mu_{x_1(i)}\right)^2 + \frac{1}{\tilde{\sigma}^2_{x_2(i)}}\left(\ln x_2-\tilde{\mu}_{x_2(i)}\right)^2 \\ &- \frac{2\tilde{\rho}_{x_1,x_2(i)}}{\sigma_{x_1(i)}\tilde{\sigma}_{x_2(i)}}\left(x_1-\mu_{x_1(i)}\right)\left(\ln x_2-\tilde{\mu}_{x_2(i)}\right)\Bigg];\end{aligned}$$

where the ith component mean of x_1 is $\mu_{x_1(i)}$, the ith component mean of $\ln x_2$ is $\tilde{\mu}_{x_2(i)}$, the ith component standard deviation of x_1 is $\sigma_{x_1(i)}$, the ith component standard deviation of

$\ln x_2$ is $\tilde{\sigma}_{x_2(i)}$, and the ith component correlation of x_1 and $\ln x_2$ is $\tilde{\rho}_{x_1,x_2(i)}$.

The third type of bivariate PDF used in the equation set is a bivariate lognormal distribution, meaning that the individual marginal for each of x_1 and x_2 is a lognormal distribution. The functional form of this type of PDF is given by:

$$P_{LL(i)}\left(x_1, x_2\right) = \frac{\exp\left\{-\frac{1}{2}\lambda_{LL}\right\}}{2\pi\tilde{\sigma}_{x_1(i)}\tilde{\sigma}_{x_2(i)}\left(1-\tilde{\rho}^2_{x_1,x_2(i)}\right)^{\frac{1}{2}} x_1 x_2}; \quad \text{where} \tag{S6}$$

$$\begin{aligned}\lambda_{LL} = \frac{1}{1-\tilde{\rho}^2_{x_1,x_2(i)}} \Bigg[& \frac{1}{\tilde{\sigma}^2_{x_1(i)}}\left(\ln x_1 - \tilde{\mu}_{x_1(i)}\right)^2 + \frac{1}{\tilde{\sigma}^2_{x_2(i)}}\left(\ln x_2 - \tilde{\mu}_{x_2(i)}\right)^2 \\ & - \frac{2\tilde{\rho}_{x_1,x_2(i)}}{\tilde{\sigma}_{x_1(i)}\tilde{\sigma}_{x_2(i)}}\left(\ln x_1 - \tilde{\mu}_{x_1(i)}\right)\left(\ln x_2 - \tilde{\mu}_{x_2(i)}\right)\Bigg];\end{aligned}$$

where the ith component mean of $\ln x_1$ is $\tilde{\mu}_{x_1(i)}$, the ith component mean of $\ln x_2$ is $\tilde{\mu}_{x_2(i)}$, the ith component standard deviation of $\ln x_1$ is $\tilde{\sigma}_{x_1(i)}$, the ith component standard deviation of $\ln x_2$ is $\tilde{\sigma}_{x_2(i)}$, and the ith component correlation of $\ln x_1$ and $\ln x_2$ is $\tilde{\rho}_{x_1,x_2(i)}$.

S4 Functional Form of Single-Variable PDFs

There are two types of single-variable (univariate) PDFs used in the equation set. The first one is a normal distribution. The functional form of this type of PDF is given by:

$$P_{N(i)}\left(x\right) = \frac{1}{\left(2\pi\right)^{\frac{1}{2}}\sigma_{x(i)}}\exp\left\{\frac{-\left(x-\mu_{x(i)}\right)^2}{2\,\sigma^2_{x(i)}}\right\}; \tag{S7}$$

where the ith component mean of x is $\mu_{x(i)}$ and the ith component standard deviation of x is $\sigma_{x(i)}$. The second type of univariate PDF used in this equation set is a lognormal distribution. If the natural logarithm was taken for every point in a lognormal distribution, the resulting distribution would be a normal distribution. The functional form of this type of PDF is given by:

$$P_{L(i)}\left(x\right)=\frac{1}{\left(2\pi\right)^{\frac{1}{2}}\,\tilde{\sigma}_{x(i)}\;x}\exp\left\{\frac{-\left(\ln x-\tilde{\mu}_{x(i)}\right)^{2}}{2\;\tilde{\sigma}_{x(i)}^{2}}\right\};\tag{S8}$$

where the ith component mean of $\ln x$ is $\tilde{\mu}_{x(i)}$ and the ith component standard deviation of $\ln x$ is $\tilde{\sigma}_{x(i)}$.

S5 Quadrivariate PDF Integrals of Covariance Form

The integrals of the general form

$$G_{QC} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}\Big(x_1 - C_1\Big)\Big(x_2^{\alpha}\left(H\left(-x_2\right)\right)^{\alpha} x_3^{\beta} x_4^{\gamma} - C_2\Big)$$
$$\times P_{NNLL}\left(x_1, x_2, x_3, x_4\right) \mathrm{d}x_4 \, \mathrm{d}x_3 \, \mathrm{d}x_2 \, \mathrm{d}x_1$$

are referred to as quadrivariate PDF integrals of covariance form. Both C_1 and C_2 are constants, and when they both represent the appropriate overall mean values, the resulting integral is a covariance. Additionally, the conditions on α are $\alpha > 0$ and $(-1)^{\alpha}$ must not be a complex number. The quadrivariate PDF, $P_{NNLL}\left(x_1, x_2, x_3, x_4\right)$, is a normal-normal-lognormal-lognormal PDF, meaning that the individual marginals of both x_1 and x_2 are normal distributions and the individual marginals of both x_3 and x_4 are lognormal distributions. The Heaviside step function is denoted $H\left(x\right)$. The above integral has 16 sub-forms. When one or more of the variables is constant (has a standard deviation of 0), the integral simplifies and reduces.

In the solutions below, denoted G_{QC}, μ_{x_1} and σ_{x_1} denote the mean and standard deviation of x_1 in the quadrivariate PDF, μ_{x_2} and σ_{x_2} denote the mean and standard deviation of x_2 in the quadrivariate PDF, μ_{x_3} and σ_{x_3} denote the mean and standard deviation of x_3 in the quadrivariate PDF, and μ_{x_4} and σ_{x_4} denote the mean and standard deviation of x_4 in the quadrivariate PDF. For lognormal variates, $\tilde{\mu}_{x_3}$ and $\tilde{\sigma}_{x_3}$ denote the mean and standard deviation of $\ln x_3$ in the quadrivariate PDF, while $\tilde{\mu}_{x_4}$ and $\tilde{\sigma}_{x_4}$ denote the mean and standard deviation of $\ln x_4$ in the quadrivariate PDF. The correlation of x_1 and x_2 is denoted ρ_{x_1,x_2}, the correlation of x_1 and $\ln x_3$ is denoted $\tilde{\rho}_{x_1,x_3}$, the correlation of x_1 and $\ln x_4$ is denoted $\tilde{\rho}_{x_1,x_4}$, the correlation of x_2 and $\ln x_3$ is denoted $\tilde{\rho}_{x_2,x_3}$, the correlation of x_2 and $\ln x_4$ is denoted $\tilde{\rho}_{x_2,x_4}$, and the correlation of $\ln x_3$ and $\ln x_4$ is denoted $\tilde{\rho}_{x_3,x_4}$. The gamma function is denoted $\Gamma\left(x\right)$ and the parabolic cylinder function of order ν is denoted $D_{\nu}\left(x\right)$.

When x_1, x_2, x_3, and x_4 all vary ($\sigma_{x_1} > 0$, $\sigma_{x_2} > 0$, $\sigma_{x_3} > 0$, and $\sigma_{x_4} > 0$), the solution is

$$\begin{aligned} G_{QC} = \frac{1}{\sqrt{2\pi}} \left(-\sigma_{x_2}\right)^{\alpha} \exp &\left\{ \tilde{\mu}_{x_3}\beta + \tilde{\mu}_{x_4}\gamma + \frac{1}{2}\left(1-\tilde{\rho}_{x_2,x_3}^2\right)\tilde{\sigma}_{x_3}^2\beta^2 \right. \\ &\left. + \frac{1}{2}\left(1-\tilde{\rho}_{x_2,x_4}^2\right)\tilde{\sigma}_{x_4}^2\gamma^2 + \left(\tilde{\rho}_{x_3,x_4} - \tilde{\rho}_{x_2,x_3}\tilde{\rho}_{x_2,x_4}\right)\tilde{\sigma}_{x_3}\beta\tilde{\sigma}_{x_4}\gamma \right\} \\ &\times \exp\left\{ \frac{1}{4}\varsigma^2 - \frac{\mu_{x_2}}{\sigma_{x_2}}\varsigma + \frac{1}{2}\frac{\mu_{x_2}^2}{\sigma_{x_2}^2} \right\} \\ &\times \Bigg(-\rho_{x_1,x_2}\sigma_{x_1}\Gamma\left(\alpha+2\right)D_{-(\alpha+2)}\left(\varsigma\right) \\ &\quad + \bigg(\mu_{x_1} - C_1 - \frac{\mu_{x_2}}{\sigma_{x_2}}\rho_{x_1,x_2}\sigma_{x_1} + \left(\tilde{\rho}_{x_1,x_3} - \rho_{x_1,x_2}\tilde{\rho}_{x_2,x_3}\right)\sigma_{x_1}\tilde{\sigma}_{x_3}\beta \\ &\qquad + \left(\tilde{\rho}_{x_1,x_4} - \rho_{x_1,x_2}\tilde{\rho}_{x_2,x_4}\right)\sigma_{x_1}\tilde{\sigma}_{x_4}\gamma \bigg)\Gamma\left(\alpha+1\right)D_{-(\alpha+1)}\left(\varsigma\right) \Bigg) \\ &- C_2\left(\mu_{x_1} - C_1\right); \end{aligned} \tag{S9}$$

where $\quad \varsigma = \dfrac{\mu_{x_2}}{\sigma_{x_2}} + \tilde{\rho}_{x_2,x_3}\tilde{\sigma}_{x_3}\beta + \tilde{\rho}_{x_2,x_4}\tilde{\sigma}_{x_4}\gamma.$

There are four sub-forms that contain one constant variable. When x_1 is constant, but x_2, x_3, and x_4 vary, the solution is

$$\begin{aligned} G_{QC} = \frac{1}{\sqrt{2\pi}} &\left(\mu_{x_1} - C_1\right)\left(-\sigma_{x_2}\right)^{\alpha} \\ &\times \exp\left\{ \tilde{\mu}_{x_3}\beta + \tilde{\mu}_{x_4}\gamma + \frac{1}{2}\left(1-\tilde{\rho}_{x_2,x_3}^2\right)\tilde{\sigma}_{x_3}^2\beta^2 \right. \\ &\qquad \left. + \frac{1}{2}\left(1-\tilde{\rho}_{x_2,x_4}^2\right)\tilde{\sigma}_{x_4}^2\gamma^2 + \left(\tilde{\rho}_{x_3,x_4} - \tilde{\rho}_{x_2,x_3}\tilde{\rho}_{x_2,x_4}\right)\tilde{\sigma}_{x_3}\beta\tilde{\sigma}_{x_4}\gamma \right\} \\ &\times \exp\left\{ \frac{1}{4}\varsigma^2 - \frac{\mu_{x_2}}{\sigma_{x_2}}\varsigma + \frac{1}{2}\frac{\mu_{x_2}^2}{\sigma_{x_2}^2} \right\}\Gamma\left(\alpha+1\right)D_{-(\alpha+1)}\left(\varsigma\right) \\ &- C_2\left(\mu_{x_1} - C_1\right); \end{aligned} \tag{S10}$$

where $\quad \varsigma = \dfrac{\mu_{x_2}}{\sigma_{x_2}} + \tilde{\rho}_{x_2,x_3}\tilde{\sigma}_{x_3}\beta + \tilde{\rho}_{x_2,x_4}\tilde{\sigma}_{x_4}\gamma.$

When x_2 is constant, but x_1, x_3, and x_4 vary, the solution is

$$G_{QC} = \begin{cases} \mu_{x_2}^{\alpha}\left(\mu_{x_1} - C_1 + \tilde{\rho}_{x_1,x_3}\sigma_{x_1}\tilde{\sigma}_{x_3}\beta + \tilde{\rho}_{x_1,x_4}\sigma_{x_1}\tilde{\sigma}_{x_4}\gamma\right) \\ \times \exp\left\{\tilde{\mu}_{x_3}\beta + \tilde{\mu}_{x_4}\gamma + \frac{1}{2}\tilde{\sigma}_{x_3}^2\beta^2 + \frac{1}{2}\tilde{\sigma}_{x_4}^2\gamma^2 + \tilde{\rho}_{x_3,x_4}\tilde{\sigma}_{x_3}\beta\tilde{\sigma}_{x_4}\gamma\right\} \\ -C_2\left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} \leq 0; \text{ and} \\ \\ -C_2\left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} > 0. \end{cases} \tag{S11}$$

When x_3 is constant, but x_1, x_2, and x_4 vary, the solution is

$$\begin{aligned} G_{QC} = & \frac{1}{\sqrt{2\pi}}\left(-\sigma_{x_2}\right)^{\alpha}\mu_{x_3}^{\beta}\exp\left\{\tilde{\mu}_{x_4}\gamma + \frac{1}{2}\tilde{\sigma}_{x_4}^2\gamma^2 - \frac{1}{4}\varsigma^2\right\} \\ & \times\Bigg(-\rho_{x_1,x_2}\sigma_{x_1}\Gamma\left(\alpha+2\right)D_{-(\alpha+2)}\left(\varsigma\right) \\ & \quad + \left(\mu_{x_1} - C_1 - \frac{\mu_{x_2}}{\sigma_{x_2}}\rho_{x_1,x_2}\sigma_{x_1} + \left(\tilde{\rho}_{x_1,x_4} - \rho_{x_1,x_2}\tilde{\rho}_{x_2,x_4}\right)\sigma_{x_1}\tilde{\sigma}_{x_4}\gamma\right) \\ & \quad\quad \times\Gamma\left(\alpha+1\right)D_{-(\alpha+1)}\left(\varsigma\right)\Bigg) \\ & - C_2\left(\mu_{x_1} - C_1\right); \end{aligned} \tag{S12}$$

where $\quad \varsigma = \dfrac{\mu_{x_2}}{\sigma_{x_2}} + \tilde{\rho}_{x_2,x_4}\tilde{\sigma}_{x_4}\gamma.$

When x_4 is constant, but x_1, x_2, and x_3 vary, the solution is

$$
\begin{aligned}
G_{QC} = & \frac{1}{\sqrt{2\pi}} \left(-\sigma_{x_2}\right)^{\alpha} \mu_{x_4}^{\gamma} \exp\left\{\tilde{\mu}_{x_3}\beta + \frac{1}{2}\tilde{\sigma}_{x_3}^{2}\beta^{2} - \frac{1}{4}\varsigma^{2}\right\} \\
& \times \Bigg(- \rho_{x_1,x_2}\sigma_{x_1}\Gamma\left(\alpha+2\right) D_{-(\alpha+2)}\left(\varsigma\right) \\
& \quad + \left(\mu_{x_1} - C_1 - \frac{\mu_{x_2}}{\sigma_{x_2}}\rho_{x_1,x_2}\sigma_{x_1} + \left(\tilde{\rho}_{x_1,x_3} - \rho_{x_1,x_2}\tilde{\rho}_{x_2,x_3}\right)\sigma_{x_1}\tilde{\sigma}_{x_3}\beta\right) \\
& \quad\quad \times \Gamma\left(\alpha+1\right) D_{-(\alpha+1)}\left(\varsigma\right) \Bigg) \\
& - C_2\left(\mu_{x_1} - C_1\right);
\end{aligned}
\tag{S13}
$$

where $\quad \varsigma = \dfrac{\mu_{x_2}}{\sigma_{x_2}} + \tilde{\rho}_{x_2,x_3}\tilde{\sigma}_{x_3}\beta.$

There are six sub-forms that contain two constant variables. When both x_1 and x_2 are constant, but both x_3 and x_4 vary, the solution is

$$
G_{QC} = \begin{cases}
\left(\mu_{x_1} - C_1\right)\mu_{x_2}^{\alpha} \exp\left\{\tilde{\mu}_{x_3}\beta + \tilde{\mu}_{x_4}\gamma + \frac{1}{2}\tilde{\sigma}_{x_3}^{2}\beta^{2} + \frac{1}{2}\tilde{\sigma}_{x_4}^{2}\gamma^{2} + \tilde{\rho}_{x_3,x_4}\tilde{\sigma}_{x_3}\beta\tilde{\sigma}_{x_4}\gamma\right\} & \\
-C_2\left(\mu_{x_1} - C_1\right), & \text{when } \mu_{x_2} \le 0; \text{ and} \\
 & \\
-C_2\left(\mu_{x_1} - C_1\right), & \text{when } \mu_{x_2} > 0.
\end{cases}
\tag{S14}
$$

When both x_1 and x_3 are constant, but both x_2 and x_4 vary, the solution is

$$
\begin{aligned}
G_{QC} = & \frac{1}{\sqrt{2\pi}}\left(\mu_{x_1} - C_1\right)\left(-\sigma_{x_2}\right)^{\alpha}\mu_{x_3}^{\beta} \exp\left\{\tilde{\mu}_{x_4}\gamma + \frac{1}{2}\tilde{\sigma}_{x_4}^{2}\gamma^{2} - \frac{1}{4}\varsigma^{2}\right\}\Gamma\left(\alpha+1\right) D_{-(\alpha+1)}\left(\varsigma\right) \\
& - C_2\left(\mu_{x_1} - C_1\right);
\end{aligned}
\tag{S15}
$$

where $\quad \varsigma = \dfrac{\mu_{x_2}}{\sigma_{x_2}} + \tilde{\rho}_{x_2,x_4}\tilde{\sigma}_{x_4}\gamma.$

When both x_1 and x_4 are constant, but both x_2 and x_3 vary, the solution is

$$\begin{aligned} G_{QC} = {} & \frac{1}{\sqrt{2\pi}} \left(\mu_{x_1} - C_1\right) \left(-\sigma_{x_2}\right)^{\alpha} \mu_{x_4}^{\gamma} \exp\left\{ \tilde{\mu}_{x_3}\beta + \frac{1}{2}\tilde{\sigma}_{x_3}^2\beta^2 - \frac{1}{4}\varsigma^2 \right\} \Gamma\left(\alpha+1\right) D_{-(\alpha+1)}\left(\varsigma\right) \\ & - C_2\left(\mu_{x_1} - C_1\right); \end{aligned} \tag{S16}$$

where $\quad \varsigma = \dfrac{\mu_{x_2}}{\sigma_{x_2}} + \tilde{\rho}_{x_2,x_3}\tilde{\sigma}_{x_3}\beta.$

When both x_2 and x_3 are constant, but both x_1 and x_4 vary, the solution is

$$G_{QC} = \begin{cases} \mu_{x_2}^{\alpha}\mu_{x_3}^{\beta}\left(\mu_{x_1} - C_1 + \tilde{\rho}_{x_1,x_4}\sigma_{x_1}\tilde{\sigma}_{x_4}\gamma\right) \exp\left\{ \tilde{\mu}_{x_4}\gamma + \dfrac{1}{2}\tilde{\sigma}_{x_4}^2\gamma^2 \right\} \\ -C_2\left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} \le 0;\ \text{and} \\ \\ -C_2\left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} > 0. \end{cases} \tag{S17}$$

When both x_2 and x_4 are constant, but both x_1 and x_3 vary, the solution is

$$G_{QC} = \begin{cases} \mu_{x_2}^{\alpha}\mu_{x_4}^{\gamma}\left(\mu_{x_1} - C_1 + \tilde{\rho}_{x_1,x_3}\sigma_{x_1}\tilde{\sigma}_{x_3}\beta\right) \exp\left\{ \tilde{\mu}_{x_3}\beta + \dfrac{1}{2}\tilde{\sigma}_{x_3}^2\beta^2 \right\} \\ -C_2\left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} \le 0;\ \text{and} \\ \\ -C_2\left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} > 0. \end{cases} \tag{S18}$$

When both x_3 and x_4 are constant, but both x_1 and x_2 vary, the solution is

$$\begin{aligned} G_{QC} = & \frac{1}{\sqrt{2\pi}} \left(-\sigma_{x_2}\right)^{\alpha} \mu_{x_3}^{\beta} \mu_{x_4}^{\gamma} \exp\left\{ -\frac{1}{4} \frac{\mu_{x_2}^2}{\sigma_{x_2}^2} \right\} \\ & \times \Bigg(-\rho_{x_1,x_2} \sigma_{x_1} \Gamma\left(\alpha+2\right) D_{-(\alpha+2)} \left(\frac{\mu_{x_2}}{\sigma_{x_2}} \right) \\ & \quad + \left(\mu_{x_1} - C_1 - \frac{\mu_{x_2}}{\sigma_{x_2}} \rho_{x_1,x_2} \sigma_{x_1} \right) \Gamma\left(\alpha+1\right) D_{-(\alpha+1)} \left(\frac{\mu_{x_2}}{\sigma_{x_2}} \right) \Bigg) \\ & - C_2 \left(\mu_{x_1} - C_1\right). \end{aligned} \tag{S19}$$

There are four sub-forms that contain three constant variables. When x_1, x_2, and x_3 are constant, but x_4 varies, the solution is

$$G_{QC} = \begin{cases} \left(\mu_{x_1} - C_1\right) \mu_{x_2}^{\alpha} \mu_{x_3}^{\beta} \exp\left\{ \tilde{\mu}_{x_4} \gamma + \frac{1}{2} \tilde{\sigma}_{x_4}^2 \gamma^2 \right\} \\ -C_2 \left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} \leq 0; \text{ and} \\ -C_2 \left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} > 0. \end{cases} \tag{S20}$$

When x_1, x_2, and x_4 are constant, but x_2 varies, the solution is

$$G_{QC} = \begin{cases} \left(\mu_{x_1} - C_1\right) \mu_{x_2}^{\alpha} \mu_{x_4}^{\gamma} \exp\left\{ \tilde{\mu}_{x_3} \beta + \frac{1}{2} \tilde{\sigma}_{x_3}^2 \beta^2 \right\} \\ -C_2 \left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} \leq 0; \text{ and} \\ -C_2 \left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} > 0. \end{cases} \tag{S21}$$

When x_1, x_3, and x_4 are constant, but x_2 varies, the solution is

$$\begin{aligned} G_{QC} = & \frac{1}{\sqrt{2\pi}} \left(\mu_{x_1} - C_1\right) \left(-\sigma_{x_2}\right)^{\alpha} \mu_{x_3}^{\beta} \mu_{x_4}^{\gamma} \exp\left\{ -\frac{1}{4} \frac{\mu_{x_2}^2}{\sigma_{x_2}^2} \right\} \Gamma\left(\alpha+1\right) D_{-(\alpha+1)} \left(\frac{\mu_{x_2}}{\sigma_{x_2}} \right) \\ & - C_2 \left(\mu_{x_1} - C_1\right). \end{aligned} \tag{S22}$$

When x_2, x_3, and x_4 are constant, but x_1 varies, the solution is

$$G_{QC} = \begin{cases} \left(\mu_{x_1} - C_1\right)\left(\mu_{x_2}^{\alpha}\mu_{x_3}^{\beta}\mu_{x_4}^{\gamma} - C_2\right), & \text{when } \mu_{x_2} \leq 0; \text{ and} \\ -C_2\left(\mu_{x_1} - C_1\right), & \text{when } \mu_{x_2} > 0. \end{cases} \tag{S23}$$

When x_1, x_2, x_3, and x_4 are all constant ($\sigma_{x_1} = 0$, $\sigma_{x_2} = 0$, $\sigma_{x_3} = 0$, and $\sigma_{x_4} = 0$), the solution is

$$G_{QC} = \begin{cases} \left(\mu_{x_1} - C_1\right)\left(\mu_{x_2}^{\alpha}\mu_{x_3}^{\beta}\mu_{x_4}^{\gamma} - C_2\right), & \text{when } \mu_{x_2} \leq 0; \text{ and} \\ -C_2\left(\mu_{x_1} - C_1\right), & \text{when } \mu_{x_2} > 0. \end{cases} \tag{S24}$$

S6 Trivariate PDF Integrals of Covariance Form

The integrals of the general form

$$G_{TC} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} \Big(x_1 - C_1\Big)\Big(x_2^{\alpha}\left(H\left(x_2\right)\right)^{\alpha} x_3^{\beta} - C_2\Big) P_{NNL}\left(x_1, x_2, x_3\right) \mathrm{d}x_3\, \mathrm{d}x_2\, \mathrm{d}x_1$$

are referred to as trivariate PDF integrals of covariance form. Both C_1 and C_2 are constants, and when they both represent the appropriate overall mean values, the resulting integral is a covariance. The trivariate PDF, $P_{NNL}\left(x_1, x_2, x_3\right)$, is a normal-normal-lognormal PDF, meaning that the individual marginals of both x_1 and x_2 are normal distributions and the individual marginal of x_3 is a lognormal distribution. The above integral has eight sub-forms. When one or more of the variables is constant (has a standard deviation of 0), the integral simplifies and reduces.

When x_1, x_2, and x_3 all vary ($\sigma_{x_1} > 0$, $\sigma_{x_2} > 0$, and $\sigma_{x_3} > 0$), the solution, denoted G_{TC}, is

$$
\begin{aligned}
G_{TC} = & \frac{1}{\sqrt{2\pi}}\,\sigma_{x_2}^{\alpha}\exp\left\{\tilde{\mu}_{x_3}\beta+\frac{1}{2}\tilde{\sigma}_{x_3}^{2}\beta^{2}-\frac{1}{4}\varsigma^{2}\right\} \\
& \times\Bigg(\rho_{x_1,x_2}\sigma_{x_1}\Gamma\left(\alpha+2\right)D_{-(\alpha+2)}\left(-\varsigma\right) \\
& \quad+\left(\mu_{x_1}-C_1-\frac{\mu_{x_2}}{\sigma_{x_2}}\rho_{x_1,x_2}\sigma_{x_1}+\left(\tilde{\rho}_{x_1,x_3}-\rho_{x_1,x_2}\tilde{\rho}_{x_2,x_3}\right)\sigma_{x_1}\tilde{\sigma}_{x_3}\beta\right) \\
& \quad\quad\times\Gamma\left(\alpha+1\right)D_{-(\alpha+1)}\left(-\varsigma\right)\Bigg) \\
& -C_2\left(\mu_{x_1}-C_1\right);
\end{aligned}
\tag{S25}
$$

where $\quad \varsigma=\dfrac{\mu_{x_2}}{\sigma_{x_2}}+\tilde{\rho}_{x_2,x_3}\tilde{\sigma}_{x_3}\beta.$

There are three sub-forms that contain one constant variable. When x_1 is constant, but x_2 and x_3 vary, the solution is

$$
\begin{aligned}
G_{TC} = & \frac{1}{\sqrt{2\pi}}\left(\mu_{x_1}-C_1\right)\sigma_{x_2}^{\alpha}\exp\left\{\tilde{\mu}_{x_3}\beta+\frac{1}{2}\tilde{\sigma}_{x_3}^{2}\beta^{2}-\frac{1}{4}\varsigma^{2}\right\}\Gamma\left(\alpha+1\right)D_{-(\alpha+1)}\left(-\varsigma\right) \\
& -C_2\left(\mu_{x_1}-C_1\right);
\end{aligned}
\tag{S26}
$$

where $\quad \varsigma=\dfrac{\mu_{x_2}}{\sigma_{x_2}}+\tilde{\rho}_{x_2,x_3}\tilde{\sigma}_{x_3}\beta.$

When x_2 is constant, but x_1 and x_3 vary, the solution is

$$
G_{TC}=\begin{cases}
\mu_{x_2}^{\alpha}\left(\mu_{x_1}-C_1+\tilde{\rho}_{x_1,x_3}\sigma_{x_1}\tilde{\sigma}_{x_3}\beta\right)\exp\left\{\tilde{\mu}_{x_3}\beta+\frac{1}{2}\tilde{\sigma}_{x_3}^{2}\beta^{2}\right\} & \\
-C_2\left(\mu_{x_1}-C_1\right), \qquad \text{when } \mu_{x_2}\geq 0;\ \text{and} & \\
& \\
-C_2\left(\mu_{x_1}-C_1\right), \qquad \text{when } \mu_{x_2}<0. &
\end{cases}
\tag{S27}
$$

When x_3 is constant, but x_1 and x_2 vary, the solution is

$$
\begin{aligned}
G_{TC} = & \frac{1}{\sqrt{2\pi}}\, \sigma_{x_2}^{\alpha} \mu_{x_3}^{\beta} \exp\left\{ -\frac{1}{4}\frac{\mu_{x_2}^2}{\sigma_{x_2}^2} \right\} \\
& \times \Bigg(\rho_{x_1,x_2}\sigma_{x_1} \Gamma\left(\alpha+2\right) D_{-(\alpha+2)}\left(-\frac{\mu_{x_2}}{\sigma_{x_2}}\right) \\
& \quad + \left(\mu_{x_1} - C_1 - \frac{\mu_{x_2}}{\sigma_{x_2}}\rho_{x_1,x_2}\sigma_{x_1}\right) \Gamma\left(\alpha+1\right) D_{-(\alpha+1)}\left(-\frac{\mu_{x_2}}{\sigma_{x_2}}\right) \Bigg) \\
& - C_2\left(\mu_{x_1} - C_1\right).
\end{aligned} \tag{S28}
$$

There are three sub-forms that contain two constant variables. When both x_1 and x_2 are constant, but x_3 varies, the solution is

$$
G_{TC} = \begin{cases} \mu_{x_2}^{\alpha}\left(\mu_{x_1} - C_1\right) \exp\left\{ \tilde{\mu}_{x_3}\beta + \frac{1}{2}\tilde{\sigma}_{x_3}^2\beta^2 \right\} - C_2\left(\mu_{x_1} - C_1\right), \text{ when } \mu_{x_2} \geq 0; \text{ and} \\ -C_2\left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} < 0. \end{cases} \tag{S29}
$$

When both x_1 and x_3 are constant, but x_2 varies, the solution is

$$
\begin{aligned}
G_{TC} = & \frac{1}{\sqrt{2\pi}}\left(\mu_{x_1} - C_1\right) \sigma_{x_2}^{\alpha} \mu_{x_3}^{\beta} \exp\left\{ -\frac{1}{4}\frac{\mu_{x_2}^2}{\sigma_{x_2}^2} \right\} \Gamma\left(\alpha+1\right) D_{-(\alpha+1)}\left(-\frac{\mu_{x_2}}{\sigma_{x_2}}\right) \\
& - C_2\left(\mu_{x_1} - C_1\right).
\end{aligned} \tag{S30}
$$

When both x_2 and x_3 are constant, but x_1 varies, the solution is

$$
G_{TC} = \begin{cases} \left(\mu_{x_1} - C_1\right)\left(\mu_{x_2}^{\alpha}\mu_{x_3}^{\beta} - C_2\right), \qquad \text{when } \mu_{x_2} \geq 0; \text{ and} \\ -C_2\left(\mu_{x_1} - C_1\right), \qquad \text{when } \mu_{x_2} < 0. \end{cases} \tag{S31}
$$

When x_1, x_2, and x_3 are all constant ($\sigma_{x_1} = 0$, $\sigma_{x_2} = 0$, and $\sigma_{x_3} = 0$), the solution is

$$G_{TC} = \begin{cases} \left(\mu_{x_1} - C_1\right)\left(\mu_{x_2}^{\alpha}\mu_{x_3}^{\beta} - C_2\right), & \text{when } \mu_{x_2} \geq 0; \text{ and} \\ -C_2\left(\mu_{x_1} - C_1\right), \quad \text{when } \mu_{x_2} < 0. \end{cases} \tag{S32}$$

S7 Trivariate PDF Integrals of Mean Form

The integrals of the general form

$$\begin{aligned} G_{TM} &= \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} x_1^{\alpha}\left(H\left(-x_1\right)\right)^{\alpha} x_2^{\beta} x_3^{\gamma} P_{NLL}\left(x_1, x_2, x_3\right) \mathrm{d}x_3 \, \mathrm{d}x_2 \, \mathrm{d}x_1 \\ &= \int_{-\infty}^{0}\int_{0}^{\infty}\int_{0}^{\infty} x_1^{\alpha} x_2^{\beta} x_3^{\gamma} P_{NLL}\left(x_1, x_2, x_3\right) \mathrm{d}x_3 \, \mathrm{d}x_2 \, \mathrm{d}x_1 \end{aligned}$$

are referred to as trivariate PDF integrals of mean form. The conditions on α are $\alpha > 0$ and $(-1)^{\alpha}$ must not be a complex number. The trivariate PDF, $P_{NLL}\left(x_1, x_2, x_3\right)$, is a normal-lognormal-lognormal PDF, meaning that the individual marginal of x_1 is a normal distribution and the individual marginals of both x_2 and x_3 are lognormal distributions. The above integral has eight sub-forms. When one or more of the variables is constant (has a standard deviation of 0), the integral simplifies and reduces.

In the solutions below, denoted G_{TM}, μ_{x_1} and σ_{x_1} denote the mean and standard deviation of x_1 in the trivariate PDF, μ_{x_2} and σ_{x_2} denote the mean and standard deviation of x_2 in the trivariate PDF, and μ_{x_3} and σ_{x_3} denote the mean and standard deviation of x_3 in the trivariate PDF. For lognormal variates, $\tilde{\mu}_{x_2}$ and $\tilde{\sigma}_{x_2}$ denote the mean and standard deviation of $\ln x_2$ in the trivariate PDF, while $\tilde{\mu}_{x_3}$ and $\tilde{\sigma}_{x_3}$ denote the mean and standard deviation of $\ln x_3$ in the trivariate PDF. The correlation of x_1 and $\ln x_2$ is denoted $\tilde{\rho}_{x_1,x_2}$, the correlation of x_1 and $\ln x_3$ is denoted $\tilde{\rho}_{x_1,x_3}$, and the correlation of $\ln x_2$ and $\ln x_3$ is denoted $\tilde{\rho}_{x_2,x_3}$. The gamma function is denoted $\Gamma\left(x\right)$ and the parabolic cylinder function of order ν is denoted $D_{\nu}\left(x\right)$.

When x_1, x_2, and x_3 all vary ($\sigma_{x_1} > 0$, $\sigma_{x_2} > 0$, and $\sigma_{x_3} > 0$), the solution is

$$
\begin{aligned}
G_{TM} = \frac{1}{\sqrt{2\pi}} \left(-\sigma_{x_1}\right)^{\alpha} \exp\Bigg\{ & \tilde{\mu}_{x_2}\beta + \tilde{\mu}_{x_3}\gamma + \frac{1}{2}\left(1 - \tilde{\rho}_{x_1,x_2}^2\right)\tilde{\sigma}_{x_2}^2\beta^2 \\
& + \frac{1}{2}\left(1 - \tilde{\rho}_{x_1,x_3}^2\right)\tilde{\sigma}_{x_3}^2\gamma^2 + \left(\tilde{\rho}_{x_2,x_3} - \tilde{\rho}_{x_1,x_2}\tilde{\rho}_{x_1,x_3}\right)\tilde{\sigma}_{x_2}\beta\tilde{\sigma}_{x_3}\gamma \Bigg\} \\
\times \exp\left\{\frac{1}{4}\varsigma^2 - \frac{\mu_{x_1}}{\sigma_{x_1}}\varsigma + \frac{1}{2}\frac{\mu_{x_1}^2}{\sigma_{x_1}^2}\right\} & \Gamma\left(\alpha+1\right) D_{-(\alpha+1)}\left(\varsigma\right);
\end{aligned}
\tag{S33}
$$

where $\quad \varsigma = \dfrac{\mu_{x_1}}{\sigma_{x_1}} + \tilde{\rho}_{x_1,x_2}\tilde{\sigma}_{x_2}\beta + \tilde{\rho}_{x_1,x_3}\tilde{\sigma}_{x_3}\gamma.$

There are three sub-forms that contain one constant variable. When x_1 is constant, but x_2 and x_3 vary, the solution is

$$
G_{TM} = \begin{cases}
\mu_{x_1}^{\alpha} \exp\left\{\tilde{\mu}_{x_2}\beta + \tilde{\mu}_{x_3}\gamma + \frac{1}{2}\tilde{\sigma}_{x_2}^2\beta^2 + \frac{1}{2}\tilde{\sigma}_{x_3}^2\gamma^2 + \tilde{\rho}_{x_2,x_3}\tilde{\sigma}_{x_2}\beta\tilde{\sigma}_{x_3}\gamma\right\}, \\
\qquad \text{when } \mu_{x_1} \leq 0; \text{ and} \\
0, \text{ when } \mu_{x_1} > 0.
\end{cases}
\tag{S34}
$$

When x_2 is constant, but x_1 and x_3 vary, the solution is

$$
G_{TM} = \frac{1}{\sqrt{2\pi}} \left(-\sigma_{x_1}\right)^{\alpha} \mu_{x_2}^{\beta} \exp\left\{\tilde{\mu}_{x_3}\gamma + \frac{1}{2}\tilde{\sigma}_{x_3}^2\gamma^2 - \frac{1}{4}\varsigma^2\right\} \Gamma\left(\alpha+1\right) D_{-(\alpha+1)}\left(\varsigma\right);
\tag{S35}
$$

where $\quad \varsigma = \dfrac{\mu_{x_1}}{\sigma_{x_1}} + \tilde{\rho}_{x_1,x_3}\tilde{\sigma}_{x_3}\gamma.$

When x_3 is constant, but x_1 and x_2 vary, the solution is

$$
G_{TM} = \frac{1}{\sqrt{2\pi}} \left(-\sigma_{x_1}\right)^{\alpha} \mu_{x_3}^{\gamma} \exp\left\{\tilde{\mu}_{x_2}\beta + \frac{1}{2}\tilde{\sigma}_{x_2}^2\beta^2 - \frac{1}{4}\varsigma^2\right\} \Gamma\left(\alpha+1\right) D_{-(\alpha+1)}\left(\varsigma\right);
\tag{S36}
$$

where $\quad \varsigma = \dfrac{\mu_{x_1}}{\sigma_{x_1}} + \tilde{\rho}_{x_1,x_2}\tilde{\sigma}_{x_2}\beta.$

There are three sub-forms that contain two constant variables. When both x_1 and x_2 are constant, but x_3 varies, the solution is

$$G_{TM} = \begin{cases} \mu_{x_1}^{\alpha}\mu_{x_2}^{\beta}\exp\left\{\tilde{\mu}_{x_3}\gamma + \frac{1}{2}\tilde{\sigma}_{x_3}^2\gamma^2\right\}, \text{ when } \mu_{x_1} \leq 0;\ \text{ and} \\ 0,\ \text{ when } \mu_{x_1} > 0. \end{cases} \tag{S37}$$

When both x_1 and x_3 are constant, but x_2 varies, the solution is

$$G_{TM} = \begin{cases} \mu_{x_1}^{\alpha}\mu_{x_3}^{\gamma}\exp\left\{\tilde{\mu}_{x_2}\beta + \frac{1}{2}\tilde{\sigma}_{x_2}^2\beta^2\right\}, \text{ when } \mu_{x_1} \leq 0;\ \text{ and} \\ 0,\ \text{ when } \mu_{x_1} > 0. \end{cases} \tag{S38}$$

When both x_2 and x_3 are constant, but x_1 varies, the solution is

$$G_{TM} = \frac{1}{\sqrt{2\pi}}\left(-\sigma_{x_1}\right)^{\alpha}\mu_{x_2}^{\beta}\mu_{x_3}^{\gamma}\exp\left\{-\frac{1}{4}\frac{\mu_{x_1}^2}{\sigma_{x_1}^2}\right\}\Gamma\left(\alpha+1\right)D_{-(\alpha+1)}\left(\frac{\mu_{x_1}}{\sigma_{x_1}}\right). \tag{S39}$$

When x_1, x_2, and x_3 are all constant ($\sigma_{x_1} = 0$, $\sigma_{x_2} = 0$, and $\sigma_{x_3} = 0$), the solution is

$$G_{TM} = \begin{cases} \mu_{x_1}^{\alpha}\mu_{x_2}^{\beta}\mu_{x_3}^{\gamma}, \text{ when } \mu_{x_1} \leq 0;\ \text{ and} \\ 0,\ \text{ when } \mu_{x_1} > 0. \end{cases} \tag{S40}$$

S8 Bivariate PDF Integrals of Mean Form

The integrals of the general form

$$\begin{aligned} G_{BM} &= \int_{-\infty}^{\infty}\int_{0}^{\infty} x_1^{\alpha}\left(H\left(x_1\right)\right)^{\alpha}x_2^{\beta}P_{NL}\left(x_1,x_2\right)\mathrm{d}x_2\,\mathrm{d}x_1 \\ &= \int_{0}^{\infty}\int_{0}^{\infty} x_1^{\alpha}x_2^{\beta}P_{NL}\left(x_1,x_2\right)\mathrm{d}x_2\,\mathrm{d}x_1 \end{aligned}$$

are referred to as bivariate PDF integrals of mean form. The bivariate PDF, $P_{NL}(x_1, x_2)$, is a normal-lognormal PDF, meaning that the individual marginal of x_1 is a normal distribution and the individual marginal of x_2 is a lognormal distribution. The above integral has four sub-forms. When one or more of the variables is constant (has a standard deviation of 0), the integral simplifies and reduces. In the solutions below, denoted G_{BM}, the notation is the same as in Section S7.

When both x_1 and x_2 vary ($\sigma_{x_1} > 0$ and $\sigma_{x_2} > 0$), the solution is

$$G_{BM} = \frac{1}{\sqrt{2\pi}}\,\sigma_{x_1}^{\alpha} \exp\left\{\tilde{\mu}_{x_2}\beta + \frac{1}{2}\tilde{\sigma}_{x_2}^2\beta^2 - \frac{1}{4}\varsigma^2\right\}\Gamma(\alpha+1)\,D_{-(\alpha+1)}(-\varsigma)\,; \tag{S41}$$

where $\quad \varsigma = \dfrac{\mu_{x_1}}{\sigma_{x_1}} + \tilde{\rho}_{x_1,x_2}\tilde{\sigma}_{x_2}\beta.$

When x_1 is constant, but x_2 varies, the solution is

$$G_{BM} = \begin{cases} \mu_{x_1}^{\alpha} \exp\left\{\tilde{\mu}_{x_2}\beta + \dfrac{1}{2}\tilde{\sigma}_{x_2}^2\beta^2\right\}, \text{ when } \mu_{x_1} \geq 0; \text{ and} \\ 0, \text{ when } \mu_{x_1} < 0. \end{cases} \tag{S42}$$

When x_2 is constant, but x_1 varies, the solution is

$$G_{BM} = \frac{1}{\sqrt{2\pi}}\,\sigma_{x_1}^{\alpha}\mu_{x_2}^{\beta} \exp\left\{-\frac{1}{4}\frac{\mu_{x_1}^2}{\sigma_{x_1}^2}\right\}\Gamma(\alpha+1)\,D_{-(\alpha+1)}\left(-\frac{\mu_{x_1}}{\sigma_{x_1}}\right). \tag{S43}$$

When both x_1 and x_2 are constant ($\sigma_{x_1} = 0$ and $\sigma_{x_2} = 0$), the solution is

$$G_{BM} = \begin{cases} \mu_{x_1}^{\alpha}\mu_{x_2}^{\beta}, \text{ when } \mu_{x_1} \geq 0; \text{ and} \\ 0, \text{ when } \mu_{x_1} < 0. \end{cases} \tag{S44}$$

S9 Solved Integrals for Higher-Order Mixed Moments

The solutions to the quadrivariate integral and its sub-forms found in Section S5 are simplified forms of the solutions to the quadrivariate integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}\Big(x_1 - C_1\Big)^a \Big(x_2^{\alpha}\left(H\left(-x_2\right)\right)^{\alpha} x_3^{\beta} x_4^{\gamma} - C_2\Big)^b P_{NNLL}\left(x_1, x_2, x_3, x_4\right) \mathrm{d}x_4\, \mathrm{d}x_3\, \mathrm{d}x_2\, \mathrm{d}x_1,$$

where both a and b are positive integers. The solutions to the generalized higher-order quadrivariate integral and its sub-forms are found in Appendix I (Section 1) of Griffin (2016). Likewise, the solutions to the trivariate integral and its sub-forms found in Section S6 are simplified forms of the solutions to the trivariate integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\Big(x_1 - C_1\Big)^a \Big(x_2^{\alpha}\left(H\left(x_2\right)\right)^{\alpha} x_3^{\beta} - C_2\Big)^b P_{NNL}\left(x_1, x_2, x_3\right) \mathrm{d}x_3\, \mathrm{d}x_2\, \mathrm{d}x_1,$$

where, again, both a and b are positive integers. The solutions to the generalized higher-order trivariate integral and its sub-forms are found in Appendix I (Section 2) of Griffin (2016).

The higher-order integrals and their solutions are important for calculating the microphysics term in a higher-order turbulent field. For example, a model that contains a predictive equation for $\overline{r_t'^3}$ or $\overline{\theta_l'^3}$ (where r_t is total water mixing ratio and θ_l is liquid water potential temperature) would require the solutions to these higher-order integrals in order to calculate the microphysics term in the predictive equation.

References

Griffin, B. M., 2016: Improving the Subgrid-Scale Representation of Hydrometeors and Microphysical Feedback Effects Using a Multivariate PDF, Ph.D. thesis, University of Wisconsin – Milwaukee, Milwaukee, WI, Paper 1144, 165 pp., URL http://dc.uwm.edu/cgi/viewcontent.cgi?article=2149&context=etd.