# Supplementary material C: The 3-D coordinate surfaces of the OS-coordinate

## 1.1 Numerical solution of each coordinate surface

The three main steps of solving each coordinate surface of the OS-coordinate are shown in Fig. C1. First, through the basis vectors of the OS-coordinate, we can obtain the partial differential equations (PDEs) of each coordinate surface of the OS-coordinate as introduced in Supplementary material B. Here are the PDEs of each coordinate surface.

For the horizontal coordinate $x^{'}$ of the OS-coordinate,

$$\frac{\partial x^{'}}{\partial x} = \cos\left(b \cdot \theta^{'}\right), \tag{1.1}$$

$$\frac{\partial x^{'}}{\partial y} = 0, \tag{1.2}$$

$$\frac{\partial x^{'}}{\partial z} = \sin\left(b \cdot \theta^{'}\right); \tag{1.3}$$

for the horizontal coordinate $y^{'}$ of the OS-coordinate,

$$\frac{\partial y^{'}}{\partial x} = -\sin\left(b \cdot \theta^{'}\right) \cdot \sin\left(b \cdot \lambda^{'}\right), \tag{1.4}$$

$$\frac{\partial y^{'}}{\partial y} = \cos\left(b \cdot \lambda^{'}\right), \tag{1.5}$$

$$\frac{\partial y^{'}}{\partial z} = \cos\left(b \cdot \theta^{'}\right) \cdot \sin\left(b \cdot \lambda^{'}\right); \tag{1.6}$$

for the vertical coordinate $\sigma$ of the OS-coordinate,

$$\frac{\partial \sigma}{\partial x} = -\sin\left(b \cdot \theta^{'}\right) \cdot \cos\left(b \cdot \lambda^{'}\right), \tag{1.7}$$

$$\frac{\partial \sigma}{\partial y} = -\sin\left(b \cdot \lambda^{'}\right), \tag{1.8}$$

$$\frac{\partial \sigma}{\partial z} = \cos\left(b \cdot \theta^{'}\right) \cdot \cos\left(b \cdot \lambda^{'}\right). \tag{1.9}$$

Second, we discretize those PDEs to obtain the linear algebraic equations (LAEs) as follows:

$$AX^{'} = B\left(f_1, g_1, h_1\right), \tag{1.10}$$

$$AY^{'} = B\left(f_2, g_2, h_2\right), \tag{1.11}$$

$$A\sigma^{'} = B\left(f_3, g_3, h_3\right), \tag{1.12}$$

where, $f_i$, $g_i$ and $h_i$ (i = 1, 2 and 3) represent the right hand side of Eqs. (1.1)-(1.3), (1.4)-(1.6), and (1.7)-(1.9), respectively.

Third, we solve these LAEs using least squares method and conjugate gradient method to obtain numerical solution of each coordinate surface of the OS-coordinate, which provides the position of every point on each coordinate surface. Figure C2 illustrates the three coordinate surfaces of the OS-coordinate solved by Eqs. (1.10)-(1.12); Fig. C3 manifests that the σ-levels in the OS-coordinate are smoother with increasing height.

Finally, we calculate the orthogonality of the points on these coordinate surfaces to exam the accuracy of these numerical solutions at the same height. We first calculate the normal vector of each coordinate surface using the points on the surface, and then via these three normal vectors we calculate the angle between every pair of the vectors. The results of these angles between each coordinate surface are shown in Fig. C4. Due to the first-order of forward scheme we used to obtain the Eqs. (1.10)-(1.12), the angles are mostly near 70 degrees. This can be improved in many ways, such as using the high-order schemes to obtain the LAEs of each coordinate surface, and using other high-order optimization method to solve the LAEs.

## 1.2 Figures

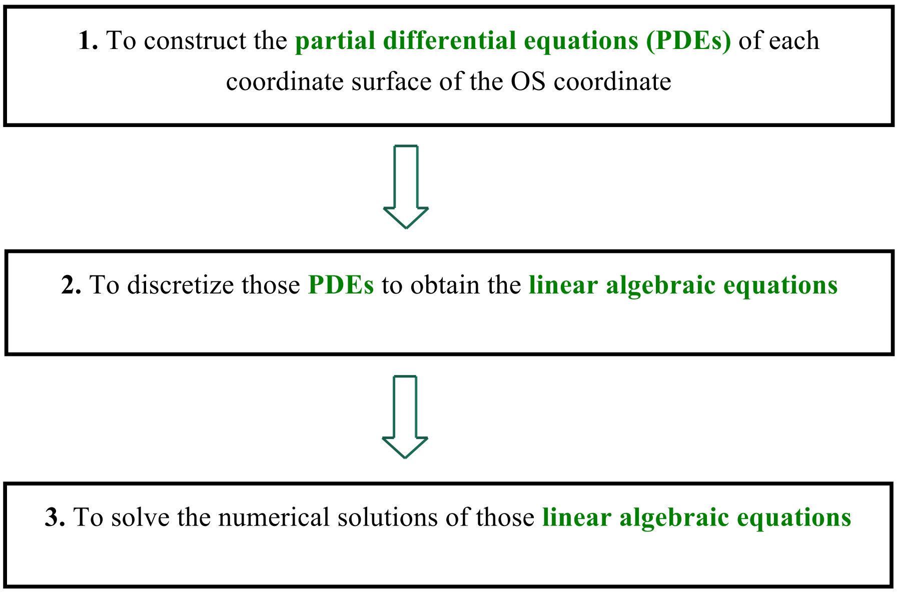


Figure C1. The three steps of solving each coordinate surface of the OS coordinate.

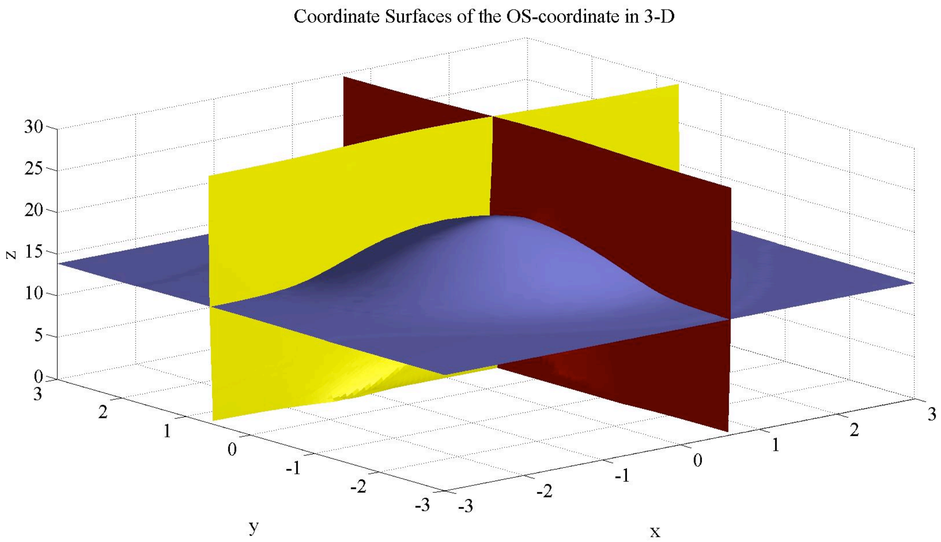


Figure C2. The three coordinate surfaces of the OS-coordinate in 3-D. Purple, red and yellow curves represent the σ-, $x^{'}$- and $y^{'}$-coordinate surfaces, respectively.

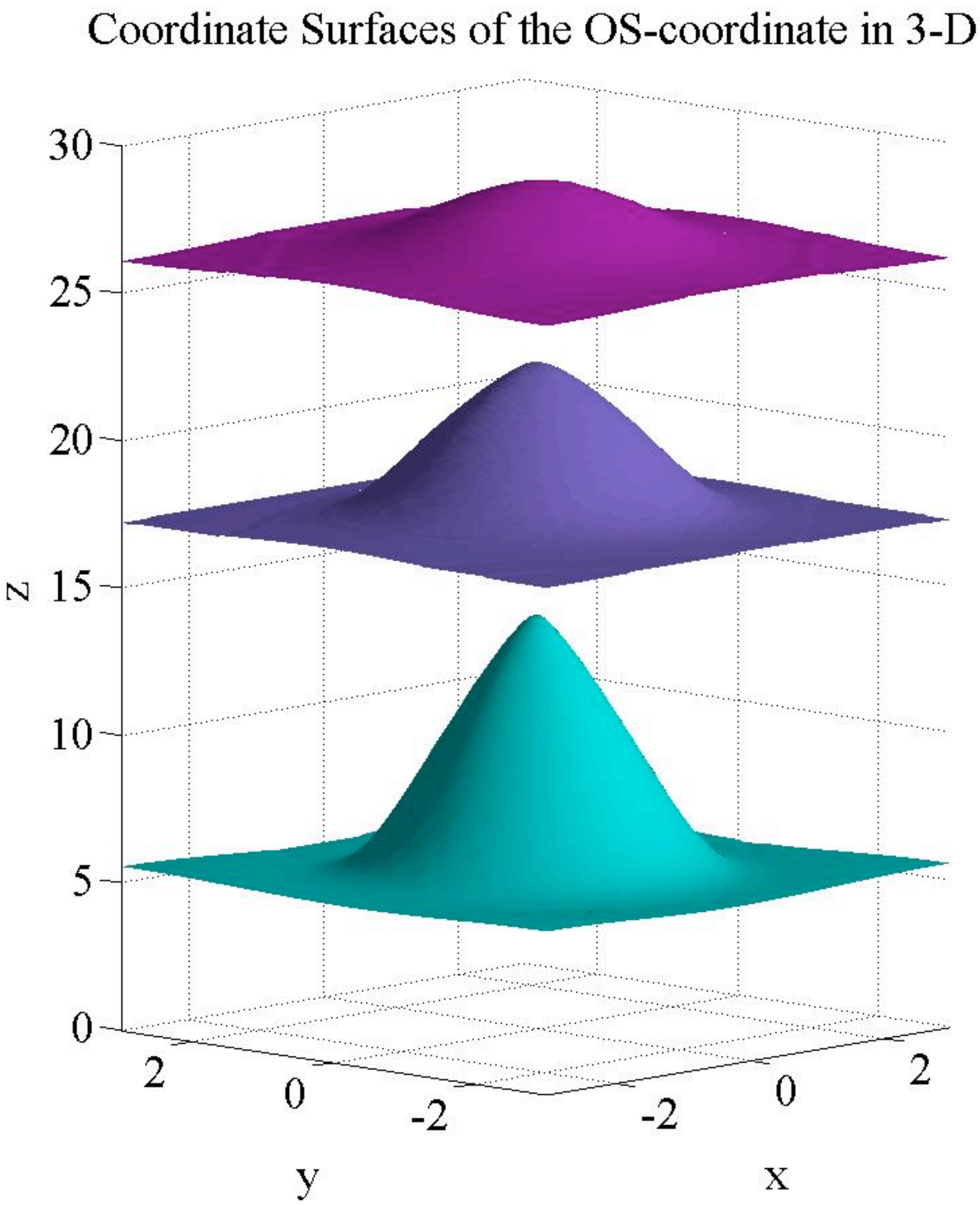


Figure C3. Vertical variation of the σ-levels in the OS-coordinate. Different colors represent different σ-levels.

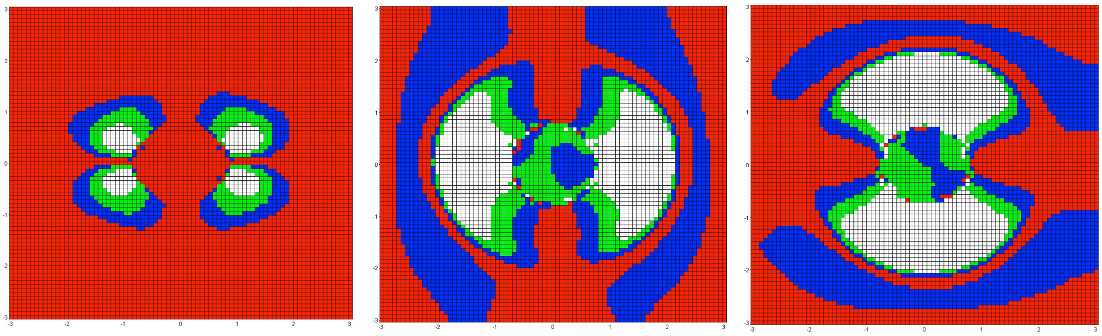

Figure C4. The orthogonality of the points on the coordinate surfaces of the OS-coordinate at a constant height. The left, middle, and right panels represent the angels between the $x^{'}$ and $y^{'}$ direction, $x^{'}$ and σ direction, and $y^{'}$ and σ direction calculated from the points on each coordinate surface, respectively. Red represents the angles range from 80-100$^{o}$, blue is for 70-80$^{o}$ or 100-110$^{o}$, and green is for 60-70$^{o}$ or 110-120$^{o}$, white is for other angels.