Supplementary material B: The transformation of the OS-coordinate

1.1 Definition of each coordinate

Since the definition of each coordinate is to clarify the "coordinate transformation" from one coordinate to the other, such as the definition of σ, we use a "cross-point way" (defined below) to solve the coordinate transformation between the z-coordinate and the OS-coordinate, instead of directly defining each coordinate in the OS-coordinate.

We use the expressions of the basis vectors of the OS-coordinate in Supplementary material A to obtain the expression of each coordinate. First, let each coordinate of the OS-coordinate be

$$x^{'} = x^{'}\left(x, y, z\right), \tag{1.1}$$

$$y^{'} = y^{'}\left(x, y, z\right), \tag{1.2}$$

$$\sigma = \sigma\left(x, y, z\right), \tag{1.3}$$

where x, y and z are the coordinates of the z-coordinate, and $x^{'}$, $y^{'}$ and σ are the coordinates of the OS-coordinate. Since the OS-coordinate is orthogonal, namely, its covariant and contravariant bases are the same, we use the contravariant basis vectors as an example. These definitions are given as follows:

$$\boldsymbol{e}^1 = \frac{\partial x^{'}}{\partial x}\boldsymbol{i} + \frac{\partial x^{'}}{\partial y}\boldsymbol{j} + \frac{\partial x^{'}}{\partial z}\boldsymbol{k}, \tag{1.4}$$

$$\boldsymbol{e}^2 = \frac{\partial y^{'}}{\partial x}\boldsymbol{i} + \frac{\partial y^{'}}{\partial y}\boldsymbol{j} + \frac{\partial y^{'}}{\partial z}\boldsymbol{k}, \tag{1.5}$$

$$\boldsymbol{e}^3 = \frac{\partial \sigma}{\partial x}\boldsymbol{i} + \frac{\partial \sigma}{\partial y}\boldsymbol{j} + \frac{\partial \sigma}{\partial z}\boldsymbol{k}. \tag{1.6}$$

By comparing Eqs. (1.4)-(1.6) with the expressions of the basis vectors of the OS-coordinate, we can obtain the partial differential equations (PDEs) of each coordinate of the OS-coordinate. Then by solving these PDEs, we obtain the expression of each coordinate of the OS-coordinate.

Since solving these PDEs of each coordinate of the OS-coordinate is difficult, we use an alternative way to define the coordinate transformation from the *z*-coordinate to the OS-coordinate, namely, to solve the coordinate values at a grid point using the OS-coordinate and then solve them again using the *z*-coordinate. We call this the "cross-point way."

1.2 Solving the coordinate lines in 2-D

In the following computations, we use the second kind of the basis vectors of the OS-coordinate as an example to elucidate how to carry out this coordinate transformation in 2-D. In addition, the transformation of the OS-coordinate in 3-D is similar to that in 2-D, and the 3-D coordinate surfaces are solved in Supplementary material C.

First, comparing (1.4)-(1.6) with the second kind of the basis vectors of the OS-coordinate, we obtain the PDEs of each coordinates as follows.

For the horizontal coordinate x' of the OS-coordinate,

$$\frac{\partial x'}{\partial x} = \cos\left(b \cdot \theta'\right), \tag{1.7}$$

$$\frac{\partial x'}{\partial y} = 0, \tag{1.8}$$

$$\frac{\partial x'}{\partial z} = \sin\left(b \cdot \theta'\right); \tag{1.9}$$

for the horizontal coordinate y' of the OS-coordinate,

$$\frac{\partial y'}{\partial x} = -\sin\left(b \cdot \theta'\right) \cdot \sin\left(b \cdot \lambda'\right), \tag{1.10}$$

$$\frac{\partial y'}{\partial y} = \cos\left(b \cdot \lambda'\right), \tag{1.11}$$

$$\frac{\partial y'}{\partial z} = \cos\left(b \cdot \theta'\right) \cdot \sin\left(b \cdot \lambda'\right); \tag{1.12}$$

for the vertical coordinate σ of the OS-coordinate,

$$\frac{\partial \sigma}{\partial x} = -\sin\left(b \cdot \theta'\right) \cdot \cos\left(b \cdot \lambda'\right), \tag{1.13}$$

$$\frac{\partial\sigma}{\partial y} = -\sin\left(b\cdot\lambda'\right), \tag{1.14}$$

$$\frac{\partial\sigma}{\partial z} = \cos\left(b\cdot\theta'\right)\cdot\cos\left(b\cdot\lambda'\right). \tag{1.15}$$

Solving these three sets of PDEs separately, we can obtain explicit expressions of x', y' and σ of the OS-coordinate. Now, we solve the coordinates of the OS-coordinate in 2-D, so the PDEs of coordinate x' and σ are transformed into the ordinary differential equations (ODEs) as follows.

For the coordinate x',

$$\frac{dz}{dx} = -\cot\left(b\cdot\theta'\right); \tag{1.16}$$

for the coordinate σ,

$$\frac{dz}{dx} = \tan\left(b\cdot\theta'\right). \tag{1.17}$$

Because the explicit solution of ODEs of (1.16) and (1.17) is difficult to obtain analytically, we solve it numerically; then, we obtain the pattern of the x'- and σ-coordinate lines shown in Fig. B1. Since the resolution of the x'-coordinate is high near the top of the terrain as shown in Fig. B1, the Courant-Friedrichs-Lewy (CFL) criterion should be considered in the design of the x'-coordinate, which needs to be dealt with by additional experiments in our future study.

Using these coordinate lines shown in Fig. B1, we then calculate the coordinate of every cross point in the *z*-coordinate and the OS-coordinate, respectively, which is exactly the coordinate transformation between the *z*-coordinate and the OS-coordinate. Specifically, the coordinate of every cross point is valued as $\left(x_1, z_1\right)$ in the *z*-coordinate, while the coordinate value of this point in the OS-coordinate is $\left(b_1, a_1\right)$, for the value of each coordinate line is its coordinate in the OS-coordinate.

Finally, through the cross-point way illustrated in Fig. B1, the coordinate transformation between the *z*-coordinate and the OS-coordinate is obtained (Fig. B2). Moreover, the coordinate transformation of the OS-coordinate preserves the three benefits of the classic σ-coordinate: (1) the points on the terrain in the *z*-coordinate (open black circles at the bottom of Fig. B2) are transformed into the points with their vertical coordinate being zero in the OS-coordinate (solid red squares at the bottom of Fig. B2); (2) when there is no-terrain and at the top of the model, the points in the OS-coordinate are those in the *z*-coordinate; (3) the irregular physical space (open black circles in Fig. B2) is transformed into a regular computational grid in the OS-coordinate (solid red squares in Fig. B2).

1.3 Figures

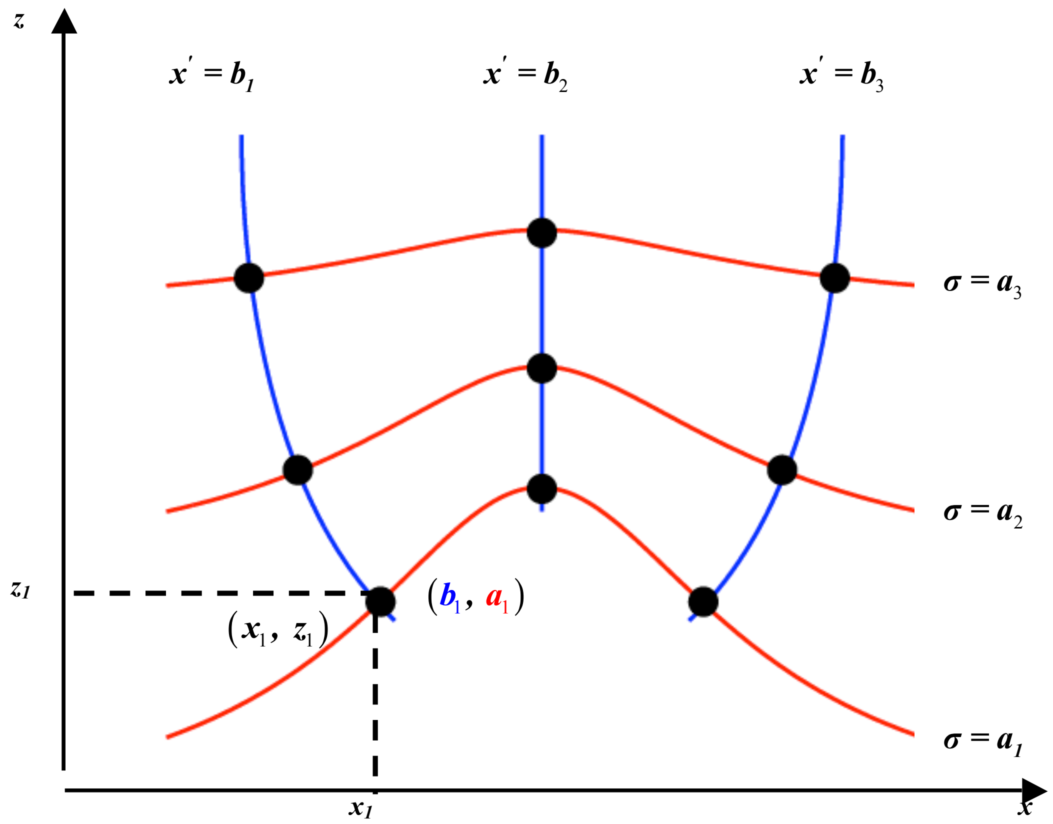

Figure B1. Schematic of the coordinate transformation between the *z*-coordinate and the OS-coordinate using the "cross-point way." The blue curves are the x'-coordinate, the red curves are the σ-coordinate, and the black dots mark where they cross each other. The *x*- and *z*-axis represent the horizontal and vertical direction of the *z*-coordinate, respectively.

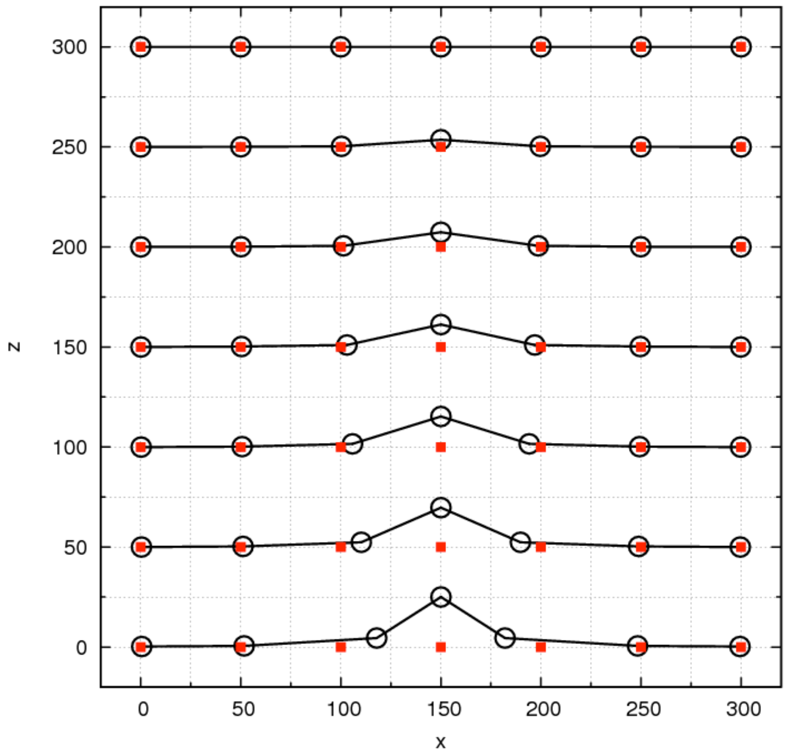

Figure B2. Transformation between the *z*-coordinate and the OS-coordinate. The solid red squares are for the OS-coordinate, and the open black circles are for the *z*-coordinate.