Supplementary material A: The 3-D coordinate rotations for solving the basis vectors of the OS-coordinate

There are four ways to rotate the basis vectors of the *z*-coordinate to obtain the basis vectors of the OS-coordinate. Specifically, there are two ways of rotations on the upslope and two on the downslope: (1) The first rotation is around the *x*-axis, and then around the rotated *y*-axis, and (2) the first rotation is around the *y*-axis, and then around the rotated *x*-axis.

The relationship between the original basis vector and the rotated one is given by

$$\begin{pmatrix} \boldsymbol{i}' \\ \boldsymbol{j}' \\ \boldsymbol{k}' \end{pmatrix} = \begin{pmatrix} \cos\alpha_1 & \cos\beta_1 & \cos\gamma_1 \\ \cos\alpha_2 & \cos\beta_2 & \cos\gamma_2 \\ \cos\alpha_3 & \cos\beta_3 & \cos\gamma_3 \end{pmatrix} \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix}, \tag{1.1}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are the basis vectors of the original coordinate; $\boldsymbol{i}'$, $\boldsymbol{j}'$ and $\boldsymbol{k}'$ are the basis vectors of the rotated coordinate; and α_i, β_i and γ_i (i = 1, 2 and 3) are the angels between the original *x*-, *y*- and *z*-axis and the corresponding rotated x'-, y'- and z'-axis. We use the space geometry to solve the rotation angles in each kind of the rotation to obtain the expression of the basis vectors of the OS-coordinate.

1.1 First kind of rotation

We solve the expression on the upslope of the terrain as shown in Fig. A1. The normal vector of the terrain is the burgundy arrow in Fig. A1 and its expression is given by

$$\boldsymbol{l} = -\frac{\partial h}{\partial x}\boldsymbol{i} - \frac{\partial h}{\partial y}\boldsymbol{j} + \boldsymbol{k}, \tag{1.2}$$

where $h = h(x, y)$ represents the terrain. On the upslope of the terrain, $\frac{\partial h}{\partial x} > 0$ and $\frac{\partial h}{\partial y} > 0$, while on the downslope of terrain, $\frac{\partial h}{\partial x} < 0$ and $\frac{\partial h}{\partial y} < 0$.

In this rotation, we first rotate the basis vectors of the *z*-coordinate around the *x*-axis and then the rotated *y*-axis (the blue arrow y_1 shown in Fig. A1), and we get two rotation angels θ^* and λ^*.

Using the space geometry, we can solve the two rotation angels respectively as follows.

For the first rotation,

$$\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\pi}{2} & \frac{\pi}{2} \\ \frac{\pi}{2} & \theta^* & \frac{\pi}{2}-\theta^* \\ \frac{\pi}{2} & \frac{\pi}{2}+\theta^* & \theta^* \end{pmatrix}; \tag{1.3}$$

for the second rotation,

$$\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} = \begin{pmatrix} \lambda^* & \frac{\pi}{2} & \frac{\pi}{2}-\lambda^* \\ \frac{\pi}{2} & 0 & \frac{\pi}{2} \\ \frac{\pi}{2}+\lambda^* & \frac{\pi}{2} & \lambda^* \end{pmatrix}, \tag{1.4}$$

where $\cos\theta^* = \dfrac{1}{\sqrt{\left(\frac{\partial h}{\partial y}\right)^2+1}}$ and $\cos\lambda^* = \dfrac{\sqrt{\left(\frac{\partial h}{\partial y}\right)^2+1}}{\sqrt{\left(\frac{\partial h}{\partial x}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2+1}}$. Substituting (1.3) into (1.1), we get the basis vectors of the coordinate $[O;\boldsymbol{x}_1,\boldsymbol{y}_1,\boldsymbol{z}_1]$ as follows:

$$\begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{y}_1 \\ \boldsymbol{z}_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta^* & \sin\theta^* \\ 0 & -\sin\theta^* & \cos\theta^* \end{pmatrix} \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix}. \tag{1.5}$$

Substituting (1.4) into (1.1), we obtain the basis vectors of the coordinate $[O;\boldsymbol{x}_2,\boldsymbol{y}_2,\boldsymbol{z}_2]$ as follows:

$$\begin{pmatrix} \boldsymbol{x}_2 \\ \boldsymbol{y}_2 \\ \boldsymbol{z}_2 \end{pmatrix} = \begin{pmatrix} \cos\lambda^* & 0 & \sin\lambda^* \\ 0 & 1 & 0 \\ -\sin\lambda^* & 0 & \cos\lambda^* \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{y}_1 \\ \boldsymbol{z}_1 \end{pmatrix}. \tag{1.6}$$

Finally, we substitute (1.5) into (1.6) to obtain the basis vectors of the coordinate $[O;\boldsymbol{x}_2,\boldsymbol{y}_2,\boldsymbol{z}_2]$, which is orthogonal and terrain-following, and its expression is given by

$$\begin{pmatrix} \boldsymbol{x}_2 \\ \boldsymbol{y}_2 \\ \boldsymbol{z}_2 \end{pmatrix} = \begin{pmatrix} \cos\lambda^* & -\sin\lambda^*\sin\theta^* & \sin\lambda^*\cos\theta^* \\ 0 & \cos\theta^* & \sin\theta^* \\ -\sin\lambda^* & -\cos\lambda^*\sin\theta^* & \cos\lambda^*\cos\theta^* \end{pmatrix} \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix}. \tag{1.7}$$

1.2 Second kind of rotation

We solve the expression of the basis vectors in the coordinate rotation on the downslope of the terrain. This rotation is first around the x-axis and then the rotated y-axis as shown in Fig. A2.

The rotation angels in these two rotations are solved as follows.

For the first rotation,

$$\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\pi}{2} & \frac{\pi}{2} \\ \frac{\pi}{2} & \theta & \frac{\pi}{2}+\theta \\ \frac{\pi}{2} & \frac{\pi}{2}-\theta & \theta \end{pmatrix}; \tag{1.8}$$

for the second rotation,

$$\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} = \begin{pmatrix} \lambda & \frac{\pi}{2} & \frac{\pi}{2}+\lambda \\ \frac{\pi}{2} & 0 & \frac{\pi}{2} \\ \frac{\pi}{2}-\lambda & \frac{\pi}{2} & \lambda \end{pmatrix}, \tag{1.9}$$

where $\cos\theta = \dfrac{1}{\sqrt{\left(\dfrac{\partial h}{\partial y}\right)^2+1}}$ and $\cos\lambda = \dfrac{\sqrt{\left(\dfrac{\partial h}{\partial y}\right)^2+1}}{\sqrt{\left(\dfrac{\partial h}{\partial x}\right)^2+\left(\dfrac{\partial h}{\partial y}\right)^2+1}}$. Using the same method of solving (1.7), we can obtain the basis vectors of the coordinate $[O;\boldsymbol{x}_2,\boldsymbol{y}_2,\boldsymbol{z}_2]$, which is orthogonal and terrain-following, and its expression is given by

$$\begin{pmatrix} \boldsymbol{x}_2 \\ \boldsymbol{y}_2 \\ \boldsymbol{z}_2 \end{pmatrix} = \begin{pmatrix} \cos\lambda & -\sin\lambda\sin\theta & -\sin\lambda\cos\theta \\ 0 & \cos\theta & -\sin\theta \\ \sin\lambda & \cos\lambda\sin\theta & \cos\lambda\cos\theta \end{pmatrix} \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix}. \tag{1.10}$$

1.3 Third kind of rotation

We solve the expression of the basis vectors in the coordinate rotation on the downslope of the terrain, which is first around the y-axis and then the rotated x-axis (Fig. A3).

The rotation angels in these two rotations are solved as follows.

For the first rotation,

$$\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} = \begin{pmatrix} \theta^{\#} & \dfrac{\pi}{2} & \dfrac{\pi}{2}+\theta^{\#} \\ \dfrac{\pi}{2} & 0 & \dfrac{\pi}{2} \\ \dfrac{\pi}{2}-\theta^{\#} & \dfrac{\pi}{2} & \theta^{\#} \end{pmatrix}; \tag{1.11}$$

for the second rotation,

$$\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{\pi}{2} & \dfrac{\pi}{2} \\ \dfrac{\pi}{2} & \lambda^{\#} & \dfrac{\pi}{2}+\lambda^{\#} \\ \dfrac{\pi}{2} & \dfrac{\pi}{2}-\lambda^{\#} & \lambda^{\#} \end{pmatrix}, \tag{1.12}$$

where $\cos\theta^{\#} = \dfrac{1}{\sqrt{\left(\dfrac{\partial h}{\partial x}\right)^2+1}}$ and $\cos\lambda^{\#} = \dfrac{\sqrt{\left(\dfrac{\partial h}{\partial x}\right)^2+1}}{\sqrt{\left(\dfrac{\partial h}{\partial x}\right)^2+\left(\dfrac{\partial h}{\partial y}\right)^2+1}}$. Using the same method of solving (1.7), we can obtain the basis vectors of the coordinate $[O;\boldsymbol{x}_2,\boldsymbol{y}_2,\boldsymbol{z}_2]$ shown in Fig. A3 as follows:

$$\begin{pmatrix} \boldsymbol{x}_2 \\ \boldsymbol{y}_2 \\ \boldsymbol{z}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta^{\#} & 0 & -\sin\theta^{\#} \\ -\sin\lambda^{\#}\sin\theta^{\#} & \cos\lambda^{\#} & -\sin\lambda^{\#}\cos\theta^{\#} \\ \cos\lambda^{\#}\sin\theta^{\#} & \sin\lambda^{\#} & \cos\lambda^{\#}\cos\theta^{\#} \end{pmatrix} \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix}. \tag{1.13}$$

1.4 Fourth kind of rotation

We solve the expression of the basis vectors in the coordinate rotation on the upslope of the terrain, which is first around the y-axis and then the rotated x-axis (Fig. A4).

The rotation angels in these two rotations are solved as follows.

For the first rotation,

$$\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} = \begin{pmatrix} \theta' & \frac{\pi}{2} & \frac{\pi}{2}-\theta' \\ \frac{\pi}{2} & 0 & \frac{\pi}{2} \\ \frac{\pi}{2}+\theta' & \frac{\pi}{2} & \theta' \end{pmatrix}; \tag{1.14}$$

for the second rotation

$$\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\pi}{2} & \frac{\pi}{2} \\ \frac{\pi}{2} & \lambda' & \frac{\pi}{2}-\lambda' \\ \frac{\pi}{2} & \frac{\pi}{2}+\lambda' & \lambda' \end{pmatrix}, \tag{1.15}$$

where $\cos\theta' = \dfrac{1}{\sqrt{\left(\frac{\partial h}{\partial x}\right)^2+1}}$, $\cos\lambda' = \dfrac{\sqrt{\left(\frac{\partial h}{\partial x}\right)^2+1}}{\sqrt{\left(\frac{\partial h}{\partial x}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2+1}}$, and $h = h(x, y)$ represents the terrain. Using the same method of solving (1.7), we can obtain the basis vectors of the coordinate $[O; \boldsymbol{x}_2, \boldsymbol{y}_2, \boldsymbol{z}_2]$ shown in Fig. A4 as follows:

$$\begin{pmatrix} \boldsymbol{x}_2 \\ \boldsymbol{y}_2 \\ \boldsymbol{z}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta' & 0 & \sin\theta' \\ -\sin\lambda'\sin\theta' & \cos\lambda' & \sin\lambda'\cos\theta' \\ -\cos\lambda'\sin\theta' & -\sin\lambda' & \cos\lambda'\cos\theta' \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix}, \tag{1.16}$$

The expressions of the basis vectors solved by all four kinds of 3-D rotations can be summarized into two sets (Table A1). Note that, since the horizontal components of these two sets of basis vectors, $\boldsymbol{i}_o$ and $\boldsymbol{j}_o$, are in the tangent plane of the terrain, and their vertical components $\boldsymbol{k}_o$ are in line with the normal vector of the terrain, these two sets of basis vectors are both terrain-following vectors except for the different directions of their horizontal components ($\boldsymbol{i}_o$ and $\boldsymbol{j}_o$).

Since the basis vectors of the z-coordinate are orthogonal, and the rotations revolved around the coordinate axes are orthogonal transformation, which keeps the original characteristics of the basis vectors, the rotated basis vectors are orthogonal. On the other hand, the z-axis of the rotated basis vectors is in line with the normal vector of the terrain, so the x-axis and y-axis of the rotated basis vectors are in the tangent plane of the terrain, which means the rotated basis vectors are terrain-following. In conclusion, all four rotated bases are both orthogonal and terrain-following.

1.5 Tables and Figures

Table A 1. Two kinds of rotated basis vectors after the 3-D rotation.

Two kinds of basis vectors	**Expressions**
The first kind	$\boldsymbol{i}_o = \boldsymbol{i}\cos\lambda - \boldsymbol{j}\sin\theta\cdot\sin\lambda - \boldsymbol{k}\cos\theta\cdot\sin\lambda$ $\boldsymbol{j}_o = \boldsymbol{j}\cos\theta - \boldsymbol{k}\sin\theta$ $\boldsymbol{k}_o = \boldsymbol{i}\sin\lambda + \boldsymbol{j}\sin\theta\cdot\cos\lambda + \boldsymbol{k}\cos\theta\cdot\cos\lambda$
The second kind	$\boldsymbol{i}_o = \boldsymbol{i}\cos\theta' + \boldsymbol{k}\sin\theta'$ $\boldsymbol{j}_o = -\boldsymbol{i}\sin\theta'\cdot\sin\lambda' + \boldsymbol{j}\cos\lambda' + \boldsymbol{k}\cos\theta'\cdot\sin\lambda'$ $\boldsymbol{k}_o = -\boldsymbol{i}\sin\theta'\cdot\cos\lambda' - \boldsymbol{j}\sin\lambda' + \boldsymbol{k}\cos\theta'\cdot\cos\lambda'$

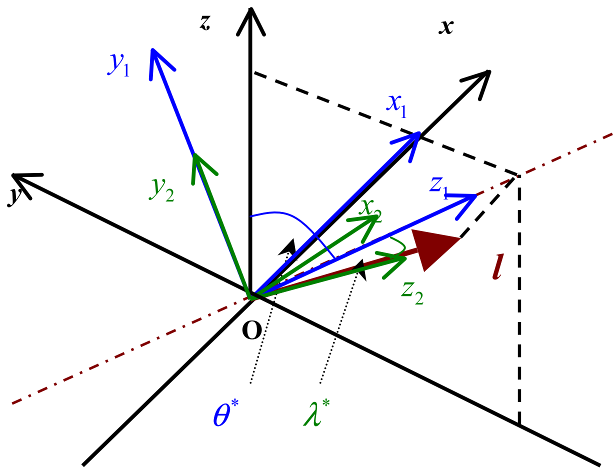

Figure A1. Schematic 3-D rotation for solving the basis vectors of the OS-coordinate on the upslope of the terrain. The burgundy arrow is the normal vector of the terrain, and the burgundy dash-dotted line is its projection on the plane *Oyz*. The black arrows are the basis vectors of the *z*-coordinate, the blue arrows are the basis vectors of the first rotated coordinate $[O; \boldsymbol{x}_1, \boldsymbol{y}_1, \boldsymbol{z}_1]$, and the green arrows are the basis vectors of the second rotated coordinate $[O; \boldsymbol{x}_2, \boldsymbol{y}_2, \boldsymbol{z}_2]$.

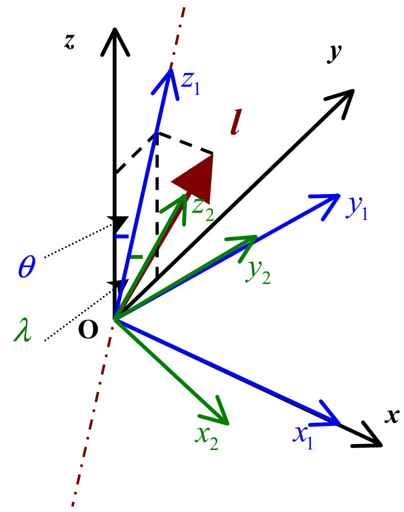

Figure A2. Same as Fig. A1, except that the rotation is on the downslope of the terrain and the first rotation is around the *x*-axis and then the rotated *y*-axis. The burgundy dash-dotted line is the projection of the normal vector of the terrain on the plane *Oyz*.

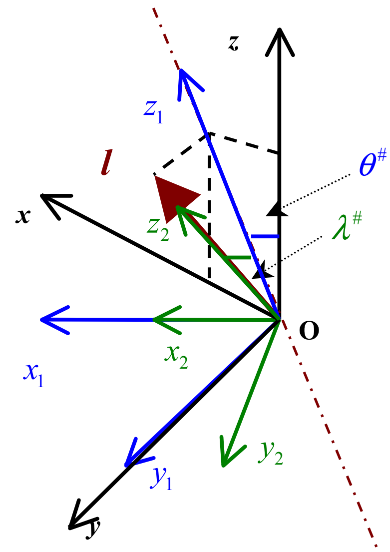

Figure A3. Same as Fig. A1, except that the rotation is on the downslope of the terrain and the first rotation is around the *y*-axis and then the rotated *x*-axis. The burgundy dash-dotted line is the projection of the normal vector of the terrain on the plane *Oxz*.

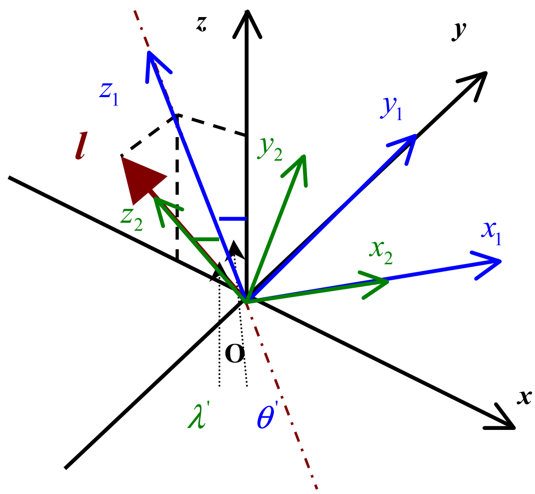

Figure A4. Same as Fig. A1, except that the rotation is on the upslope of the terrain and the first rotation is around the *y*-axis and then the rotated *x*-axis. The burgundy dash-dotted line is the projection of the normal vector of the terrain on the plane *Oxz*.