



Supplement of

MONKI: a three-dimensional Monte Carlo simulator for transfer of polarised light in planetary atmospheres

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Supplementary material

Green's functions

Definition of a Green's function

The method of Green's functions is a powerful way to solve an inhomogeneous linear boundary-value problem, that is, an equation of the form (see, e.g., Stakgold and Holst, 2011)

$$\mathcal{L}[u(x)] = q(x), \tag{S1}$$

where u and q are functions of a variable x , and $\mathcal{L}$ is a linear operator acting on x . We consider homogeneous boundary conditions,

$$\begin{aligned} u(a) &= 0, \\ u(b) &= 0, \end{aligned} \tag{S2}$$

where a and b are real values with $a < b$. The objective is to find an expression for $u(x)$.

The Green's function $G(x, x^*)$ for this boundary-value problem is defined as the solution of

$$\mathcal{L}[G(x, x^*)] = \delta(x - x^*), \tag{S3}$$

where δ is the Dirac delta function, with the same homogeneous boundary conditions,

$$\begin{aligned} G(a, x^*) &= 0, \\ G(b, x^*) &= 0. \end{aligned} \tag{S4}$$

Multiplying Eq. S3 by $q(x^*)$ and integrating over x^* gives

$$\int_a^b \mathcal{L}[G(x, x^*)] q(x^*) dx^* = \int_a^b \delta(x - x^*) q(x^*) dx^*. \tag{S5}$$

The right-hand side can be evaluated using the defining property of the Dirac delta function. On the left-hand side, the linear operator $\mathcal{L}$ can be interchanged with the integration over x^* , because $\mathcal{L}$ acts on x and not on x^* . This gives

$$\mathcal{L} \left[\int_a^b G(x, x^*) q(x^*) dx^* \right] = q(x). \tag{S6}$$

Comparing Eq. S6 with Eq. S1, and using the fact that G satisfies the same homogeneous boundary conditions, we obtain

$$u(x) = \int_a^b G(x, x^*) q(x^*) dx^*. \tag{S7}$$

Thus, once the Green's function is known, the solution can be written as a weighted integral over the source term q . The Green's function $G(x, x^*)$ can sometimes be found directly from Eq. S3 and the boundary conditions, as shown in the example below.

Example

Let us consider the boundary-value problem

$$-\alpha u''(x) = q(x), \quad (\text{S8})$$

where $u'' \equiv d^2u/dx^2$ is the second derivative of u with respect to x , and α is a positive constant. The boundary conditions are

$$u(0) = 0, \quad (\text{S9})$$

$$u(B) = 0. \quad (\text{S10})$$

Here, the linear operator is

$$\mathcal{L}[u(x)] = -\alpha u''(x). \quad (\text{S11})$$

The Green's function is defined by

$$\mathcal{L}[G(x, x^*)] = \delta(x - x^*), \quad (\text{S12})$$

or equivalently,

$$-\alpha G''(x, x^*) = \delta(x - x^*). \quad (\text{S13})$$

Hence,

$$G''(x, x^*) = -\frac{1}{\alpha} \delta(x - x^*). \quad (\text{S14})$$

Integrating once with respect to x gives

$$G'(x, x^*) = -\frac{1}{\alpha} H(x - x^*) + C_1(x^*), \quad (\text{S15})$$

where H is the Heaviside step function and C_1 may depend on x^* . Thus,

$$G'(x, x^*) = \begin{cases} C_1(x^*), & x < x^*, \\ -\frac{1}{\alpha} + C_1(x^*), & x > x^*. \end{cases} \quad (\text{S16})$$

Integrating once more gives¹⁵

$$G(x, x^*) = \begin{cases} C_1(x^*)x + C_2(x^*), & x < x^*, \\ -\frac{x - x^*}{\alpha} + C_1(x^*)x + C_2(x^*), & x \geq x^*. \end{cases} \quad (\text{S17})$$

¹⁵We include $x = x^*$ in the second branch. At $x = x^*$, both branches reduce to the same value, so $G(x, x^*)$ is continuous there, although its first derivative is discontinuous.

The boundary condition at $x = 0$ gives

$$G(0, x^*) = 0 \quad \Rightarrow \quad C_2(x^*) = 0. \quad (\text{S18})$$

The boundary condition at $x = B$ gives

$$G(B, x^*) = 0 \quad \Rightarrow \quad -\frac{B - x^*}{\alpha} + C_1(x^*)B = 0, \quad (\text{S19})$$

so that

$$C_1(x^*) = \frac{B - x^*}{\alpha B}. \quad (\text{S20})$$

The Green's function is therefore

$$G(x, x^*) = \begin{cases} \frac{x(B - x^*)}{\alpha B}, & x < x^*, \\ \frac{x^*(B - x)}{\alpha B}, & x \geq x^*. \end{cases} \quad (\text{S21})$$

The solution of the boundary-value problem is then

$$u(x) = \int_0^B G(x, x^*) q(x^*) \mathrm{d}x^*. \quad (\text{S22})$$

Adjoint transport operator identity

To verify the adjoint identity $(K[\mathbf{u}], \mathbf{g}) = (\mathbf{u}, K^+[\mathbf{g}])$ stated in Eq. G19, we start from the definition of the operator K ,

$$K[\mathbf{u}](\mathbf{x}, \boldsymbol{\omega}) = \int_{\mathcal{V}} \int_{4\pi} \mathbf{k}[(\mathbf{x}^*, \boldsymbol{\omega}') \rightarrow (\mathbf{x}, \boldsymbol{\omega})] \mathbf{u}(\mathbf{x}^*, \boldsymbol{\omega}') d\Omega' d^3\mathbf{x}^*, \quad (\text{S23})$$

and the inner product

$$(\mathbf{u}, \mathbf{v}) = \int_{\mathcal{V}} \int_{4\pi} \mathbf{u}(\mathbf{x}, \boldsymbol{\omega})^T \mathbf{v}(\mathbf{x}, \boldsymbol{\omega}) d\Omega d^3\mathbf{x}. \quad (\text{S24})$$

Then

$$\begin{aligned} (K[\mathbf{u}], \mathbf{g}) &= \int_{\mathcal{V}} \int_{4\pi} (K[\mathbf{u}](\mathbf{x}, \boldsymbol{\omega}))^T \mathbf{g}(\mathbf{x}, \boldsymbol{\omega}) d\Omega d^3\mathbf{x} \\ &= \int_{\mathcal{V}} \int_{4\pi} \left[\int_{\mathcal{V}} \int_{4\pi} \mathbf{k}[(\mathbf{x}^*, \boldsymbol{\omega}') \rightarrow (\mathbf{x}, \boldsymbol{\omega})] \mathbf{u}(\mathbf{x}^*, \boldsymbol{\omega}') d\Omega' d^3\mathbf{x}^* \right]^T \mathbf{g}(\mathbf{x}, \boldsymbol{\omega}) d\Omega d^3\mathbf{x} \\ &= \int_{\mathcal{V}} \int_{4\pi} \int_{\mathcal{V}} \int_{4\pi} \mathbf{u}(\mathbf{x}^*, \boldsymbol{\omega}')^T \mathbf{k}[(\mathbf{x}^*, \boldsymbol{\omega}') \rightarrow (\mathbf{x}, \boldsymbol{\omega})]^T \mathbf{g}(\mathbf{x}, \boldsymbol{\omega}) d\Omega' d^3\mathbf{x}^* d\Omega d^3\mathbf{x}, \end{aligned} \quad (\text{S25})$$

where we used the identity $(\mathbf{A}\mathbf{u})^T = \mathbf{u}^T \mathbf{A}^T$. Reordering the integrations gives

$$(K[\mathbf{u}], \mathbf{g}) = \int_{\mathcal{V}} \int_{4\pi} \mathbf{u}(\mathbf{x}^*, \boldsymbol{\omega}')^T \left[\int_{\mathcal{V}} \int_{4\pi} \mathbf{k}[(\mathbf{x}^*, \boldsymbol{\omega}') \rightarrow (\mathbf{x}, \boldsymbol{\omega})]^T \mathbf{g}(\mathbf{x}, \boldsymbol{\omega}) d\Omega d^3\mathbf{x} \right] d\Omega' d^3\mathbf{x}^*. \quad (\text{S26})$$

Now relabel the dummy integration variables $(\mathbf{x}^*, \boldsymbol{\omega}') \leftrightarrow (\mathbf{x}, \boldsymbol{\omega})$. Then Eq. S26 becomes

$$(K[\mathbf{u}], \mathbf{g}) = \int_{\mathcal{V}} \int_{4\pi} \mathbf{u}(\mathbf{x}, \boldsymbol{\omega})^T \left[\int_{\mathcal{V}} \int_{4\pi} \mathbf{k}[(\mathbf{x}, \boldsymbol{\omega}) \rightarrow (\mathbf{x}^*, \boldsymbol{\omega}')]^T \mathbf{g}(\mathbf{x}^*, \boldsymbol{\omega}') d\Omega' d^3\mathbf{x}^* \right] d\Omega d^3\mathbf{x}. \quad (\text{S27})$$

Recognizing the term in brackets as the definition of $K^+[\mathbf{g}]$,

$$K^+[\mathbf{g}](\mathbf{x}, \boldsymbol{\omega}) = \int_{\mathcal{V}} \int_{4\pi} \mathbf{k}[(\mathbf{x}, \boldsymbol{\omega}) \rightarrow (\mathbf{x}^*, \boldsymbol{\omega}')]^T \mathbf{g}(\mathbf{x}^*, \boldsymbol{\omega}') d\Omega' d^3\mathbf{x}^*, \quad (\text{S28})$$

we obtain

$$\begin{aligned} (K[\mathbf{u}], \mathbf{g}) &= \int_{\mathcal{V}} \int_{4\pi} \mathbf{u}(\mathbf{x}, \boldsymbol{\omega})^T K^+[\mathbf{g}](\mathbf{x}, \boldsymbol{\omega}) d\Omega d^3\mathbf{x} \\ &= (\mathbf{u}, K^+[\mathbf{g}]). \end{aligned} \quad (\text{S29})$$

Hence,

$$(K[\mathbf{u}], \mathbf{g}) = (\mathbf{u}, K^+[\mathbf{g}]), \quad (\text{S30})$$

which proves Eq. G19. It should be noted that reciprocity is not used in the proof above. The transpose in the definition of K^+ follows solely from the Stokes vector inner product.

Additional tables and figures

Table S1. Similar to Table 1, except for $(\mu_0, \varphi_0) = (0.1, 0^\circ)$.

		$\mu = 0.1$		$\mu = 0.5$		$\mu = 1.0$	
		de Haan	MONKI	de Haan	MONKI	de Haan	MONKI
$\mu_0 = 0.1$ $\varphi = 0^\circ$	I	0.522770	0.522851 $\pm$ 0.000324	0.106590	0.106604 $\pm$ 0.000067	0.026009	0.026015 $\pm$ 0.000013
	Q	0.011506	0.011495 $\pm$ 0.000045	-0.005186	-0.005189 $\pm$ 0.000015	-0.014984	-0.014986 $\pm$ 0.000008
	U	0.000000	0.000018 $\pm$ 0.000032	0.000000	0.000003 $\pm$ 0.000014	0.000000	-0.000002 $\pm$ 0.000005
	V	0.000000	-0.000000 $\pm$ 0.000003	0.000000	0.000001 $\pm$ 0.000002	0.000000	0.000000 $\pm$ 0.000001
$\mu_0 = 0.1$ $\varphi = 30^\circ$	I	0.276300	0.276336 $\pm$ 0.000153	0.083628	0.083659 $\pm$ 0.000054	0.026009	0.026015 $\pm$ 0.000013
	Q	0.034368	0.034354 $\pm$ 0.000048	0.003839	0.003847 $\pm$ 0.000017	-0.007492	-0.007491 $\pm$ 0.000006
	U	-0.016042	-0.016039 $\pm$ 0.000030	-0.014492	-0.014497 $\pm$ 0.000013	-0.012976	-0.012980 $\pm$ 0.000007
	V	0.000027	0.000028 $\pm$ 0.000003	0.000017	0.000018 $\pm$ 0.000002	0.000000	0.000000 $\pm$ 0.000001

Table S2. Similar to Table 1, except for downward light at the surface (BOA).

		$\mu = 0.1$		$\mu = 0.5$		$\mu = 1.0$	
		de Haan	MONKI	de Haan	MONKI	de Haan	MONKI
$\mu_0 = 0.5$ $\varphi = 0^\circ$	I	1.036450	1.036876 $\pm$ 0.001867	2.287330	2.287212 $\pm$ 0.001710	0.088417	0.088341 $\pm$ 0.000121
	Q	0.010451	0.010411 $\pm$ 0.000221	0.007210	0.007216 $\pm$ 0.000091	-0.023375	-0.023339 $\pm$ 0.000054
	U	0.000000	-0.000037 $\pm$ 0.000196	0.000000	0.000012 $\pm$ 0.000086	0.000000	-0.000004 $\pm$ 0.000028
	V	0.000000	0.000015 $\pm$ 0.000021	0.000000	0.000007 $\pm$ 0.000008	0.000000	0.000002 $\pm$ 0.000003
$\mu_0 = 0.5$ $\varphi = 30^\circ$	I	0.566020	0.566066 $\pm$ 0.001217	0.604840	0.604612 $\pm$ 0.000578	0.088417	0.088341 $\pm$ 0.000121
	Q	0.013634	0.013609 $\pm$ 0.000208	0.012593	0.012600 $\pm$ 0.000087	-0.011687	-0.011673 $\pm$ 0.000030
	U	-0.016244	-0.016223 $\pm$ 0.000231	-0.002307	-0.002334 $\pm$ 0.000079	0.020243	0.020210 $\pm$ 0.000053
	V	0.000084	0.000086 $\pm$ 0.000025	-0.000137	-0.000135 $\pm$ 0.000007	0.000000	0.000002 $\pm$ 0.000003

Table S3. Similar to Table S1, except for downward light at the surface (BOA).

		$\mu = 0.1$		$\mu = 0.5$		$\mu = 1.0$	
		de Haan	MONKI	de Haan	MONKI	de Haan	MONKI
$\mu_0 = 0.1$ $\varphi = 0^\circ$	I	0.191930	0.191924 $\pm$ 0.000392	0.129879	0.129944 $\pm$ 0.000106	0.022968	0.022975 $\pm$ 0.000026
	Q	0.003451	0.003448 $\pm$ 0.000062	0.000363	0.000347 $\pm$ 0.000025	-0.013424	-0.013432 $\pm$ 0.000019
	U	0.000000	-0.000007 $\pm$ 0.000052	0.000000	0.000012 $\pm$ 0.000019	0.000000	0.000003 $\pm$ 0.000008
	V	0.000000	0.000001 $\pm$ 0.000005	0.000000	0.000001 $\pm$ 0.000002	0.000000	-0.000000 $\pm$ 0.000001
$\mu_0 = 0.1$ $\varphi = 30^\circ$	I	0.086631	0.086559 $\pm$ 0.000243	0.084823	0.084868 $\pm$ 0.000107	0.022968	0.022975 $\pm$ 0.000026
	Q	0.006335	0.006333 $\pm$ 0.000059	0.005860	0.005843 $\pm$ 0.000025	-0.006712	-0.006713 $\pm$ 0.000011
	U	0.001320	0.001327 $\pm$ 0.000037	0.006928	0.006938 $\pm$ 0.000021	0.011626	0.011634 $\pm$ 0.000017
	V	-0.000010	-0.000006 $\pm$ 0.000007	-0.000062	-0.000059 $\pm$ 0.000002	0.000000	-0.000000 $\pm$ 0.000001

Table S4. Comparison of the TOA radiances computed with MONKI using 10^7 photons in backward mode with the benchmark results of de Haan et al. (1987) for an atmosphere consisting of two plane-parallel layers with a total optical thickness $b = 0.6$ (see text). The direction of the incident unpolarised light beam is $(\mu_0, \varphi_0) = (0.5, 0^\circ)$. The reported uncertainties in the MONKI results refer to twice the standard error of the mean, evaluated separately for each Stokes component using Eq. 23.

		$\mu = 0.1$		$\mu = 0.5$		$\mu = 1.0$	
		de Haan	MONKI	de Haan	MONKI	de Haan	MONKI
$\mu_0 = 0.5$ $\varphi = 0^\circ$	I	0.532950	0.533025 $\pm$ 0.000329	0.208430	0.208418 $\pm$ 0.000235	0.093680	0.093736 $\pm$ 0.000133
	Q	-0.028340	-0.028370 $\pm$ 0.000085	-0.036299	-0.036337 $\pm$ 0.000076	-0.024156	-0.024202 $\pm$ 0.000062
	U	0.000000	0.000038 $\pm$ 0.000079	0.000000	0.000041 $\pm$ 0.000056	0.000000	0.000032 $\pm$ 0.000041
	V	0.000000	-0.000000 $\pm$ 0.000006	0.000000	-0.000004 $\pm$ 0.000006	0.000000	0.000003 $\pm$ 0.000005
$\mu_0 = 0.5$ $\varphi = 30^\circ$	I	0.418140	0.418242 $\pm$ 0.000268	0.184970	0.184817 $\pm$ 0.000217	0.093680	0.093756 $\pm$ 0.000152
	Q	-0.000058	-0.000077 $\pm$ 0.000085	-0.019649	-0.019652 $\pm$ 0.000062	-0.012078	-0.012128 $\pm$ 0.000052
	U	-0.073705	-0.073111 $\pm$ 0.000084	-0.041401	-0.041354 $\pm$ 0.000083	-0.020920	-0.020920 $\pm$ 0.000057
	V	0.000106	0.000107 $\pm$ 0.000006	0.000040	0.000037 $\pm$ 0.000006	0.000000	0.000005 $\pm$ 0.000005

Table S5. Similar to Table S4, except for $(\mu_0, \varphi_0) = (0.1, 0^\circ)$.

		$\mu = 0.1$		$\mu = 0.5$		$\mu = 1.0$	
		de Haan	MONKI	de Haan	MONKI	de Haan	MONKI
$\mu_0 = 0.1$ $\varphi = 0^\circ$	I	0.522770	0.522857 $\pm$ 0.000325	0.106590	0.106499 $\pm$ 0.000121	0.026009	0.026053 $\pm$ 0.000049
	Q	0.011506	0.011484 $\pm$ 0.000047	-0.005186	-0.005194 $\pm$ 0.000032	-0.014984	-0.015008 $\pm$ 0.000038
	U	0.000000	-0.000021 $\pm$ 0.000034	0.000000	-0.000002 $\pm$ 0.000024	0.000000	0.000015 $\pm$ 0.000017
	V	0.000000	0.000000 $\pm$ 0.000002	0.000000	-0.000004 $\pm$ 0.000003	0.000000	0.000001 $\pm$ 0.000002
$\mu_0 = 0.1$ $\varphi = 30^\circ$	I	0.276300	0.276342 $\pm$ 0.000168	0.083628	0.083548 $\pm$ 0.000114	0.026009	0.026046 $\pm$ 0.000050
	Q	0.034368	0.034349 $\pm$ 0.000054	0.003839	0.003826 $\pm$ 0.000029	-0.007492	-0.007502 $\pm$ 0.000025
	U	-0.016042	-0.016057 $\pm$ 0.000036	-0.014492	-0.014488 $\pm$ 0.000032	-0.012976	-0.012988 $\pm$ 0.000029
	V	0.000027	0.000026 $\pm$ 0.000002	0.000017	0.000013 $\pm$ 0.000003	0.000000	0.000002 $\pm$ 0.000002

Table S6. Similar to Table S4, except for downward light at the surface (BOA).

		$\mu = 0.1$		$\mu = 0.5$		$\mu = 1.0$	
		de Haan	MONKI	de Haan	MONKI	de Haan	MONKI
$\mu_0 = 0.5$ $\varphi = 0^\circ$	I	1.036450	1.036884 $\pm$ 0.000626	2.287330	2.287353 $\pm$ 0.001495	0.088417	0.088405 $\pm$ 0.000156
	Q	0.010451	0.010466 $\pm$ 0.000076	0.007210	0.007229 $\pm$ 0.000070	-0.023375	-0.023392 $\pm$ 0.000064
	U	0.000000	0.000039 $\pm$ 0.000055	0.000000	-0.000027 $\pm$ 0.000058	0.000000	-0.000010 $\pm$ 0.000052
	V	0.000000	0.000007 $\pm$ 0.000008	0.000000	-0.000001 $\pm$ 0.000005	0.000000	0.000001 $\pm$ 0.000004
$\mu_0 = 0.5$ $\varphi = 30^\circ$	I	0.566020	0.566619 $\pm$ 0.000519	0.604840	0.604551 $\pm$ 0.000530	0.088417	0.088425 $\pm$ 0.000152
	Q	0.013634	0.013664 $\pm$ 0.000081	0.012593	0.012595 $\pm$ 0.000070	-0.011687	-0.011700 $\pm$ 0.000051
	U	-0.016244	-0.016226 $\pm$ 0.000064	-0.002307	-0.002325 $\pm$ 0.000065	0.020243	0.020205 $\pm$ 0.000061
	V	0.000084	0.000090 $\pm$ 0.000008	-0.000137	-0.000139 $\pm$ 0.000006	0.000000	0.000001 $\pm$ 0.000004

Table S7. Similar to Table S5, except for downward light at the surface (BOA).

		$\mu = 0.1$		$\mu = 0.5$		$\mu = 1.0$	
		de Haan	MONKI	de Haan	MONKI	de Haan	MONKI
$\mu_0 = 0.1$ $\varphi = 0^\circ$	I	0.191930	0.191814 $\pm$ 0.000243	0.129879	0.129851 $\pm$ 0.000162	0.022968	0.022924 $\pm$ 0.000054
	Q	0.003451	0.003459 $\pm$ 0.000028	0.000363	0.000366 $\pm$ 0.000027	-0.013424	-0.013411 $\pm$ 0.000033
	U	0.000000	0.000017 $\pm$ 0.000025	0.000000	-0.000026 $\pm$ 0.000025	0.000000	0.000004 $\pm$ 0.000022
	V	0.000000	0.000003 $\pm$ 0.000003	0.000000	0.000001 $\pm$ 0.000002	0.000000	0.000001 $\pm$ 0.000001
$\mu_0 = 0.1$ $\varphi = 30^\circ$	I	0.086631	0.086687 $\pm$ 0.000130	0.084823	0.084820 $\pm$ 0.000122	0.022968	0.022956 $\pm$ 0.000053
	Q	0.006335	0.006345 $\pm$ 0.000033	0.005860	0.005853 $\pm$ 0.000031	-0.006712	-0.006716 $\pm$ 0.000024
	U	0.001320	0.001339 $\pm$ 0.000025	0.006928	0.006910 $\pm$ 0.000027	0.011626	0.011611 $\pm$ 0.000032
	V	-0.000010	-0.000007 $\pm$ 0.000003	-0.000062	-0.000062 $\pm$ 0.000003	0.000000	0.000000 $\pm$ 0.000001

Table S8. Viewing and solar geometries for the 9 3D cloud test cases of Emde et al. (2018), using the angular definitions of MONKI. The azimuth angle of the sunlight is $\varphi_0 = 0^\circ$ for all cases. The solar zenith angle θ_0 is defined with respect to the local zenith. For the cases with $\vartheta > 90^\circ$ the detector is at BOA, otherwise the detector is at TOA.

No.	θ_0 [$^\circ$]	ϑ [$^\circ$]	φ [$^\circ$]
1	20	140	180
2	20	140	240
3	20	140	300
4	20	140	0
5	40	0	180
6	40	40	180
7	40	40	240
8	40	40	300
9	40	40	0

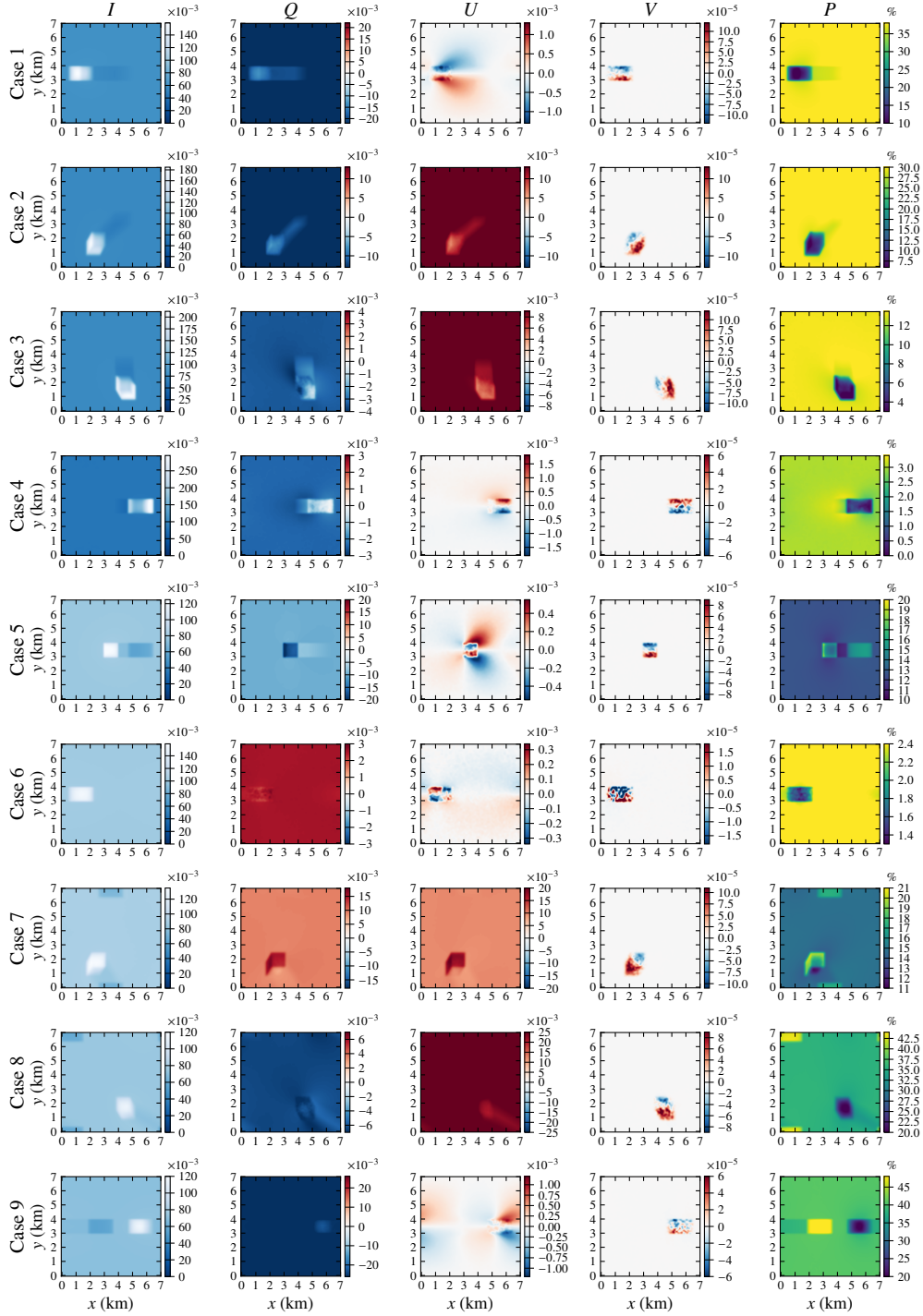


Figure S1. Similar to Fig. 7, except for all Stokes parameters I , Q , U , and V , and for degree of polarisation P (the columns), at all 9 solar and viewing geometry cases (the rows) presented in Table S8.

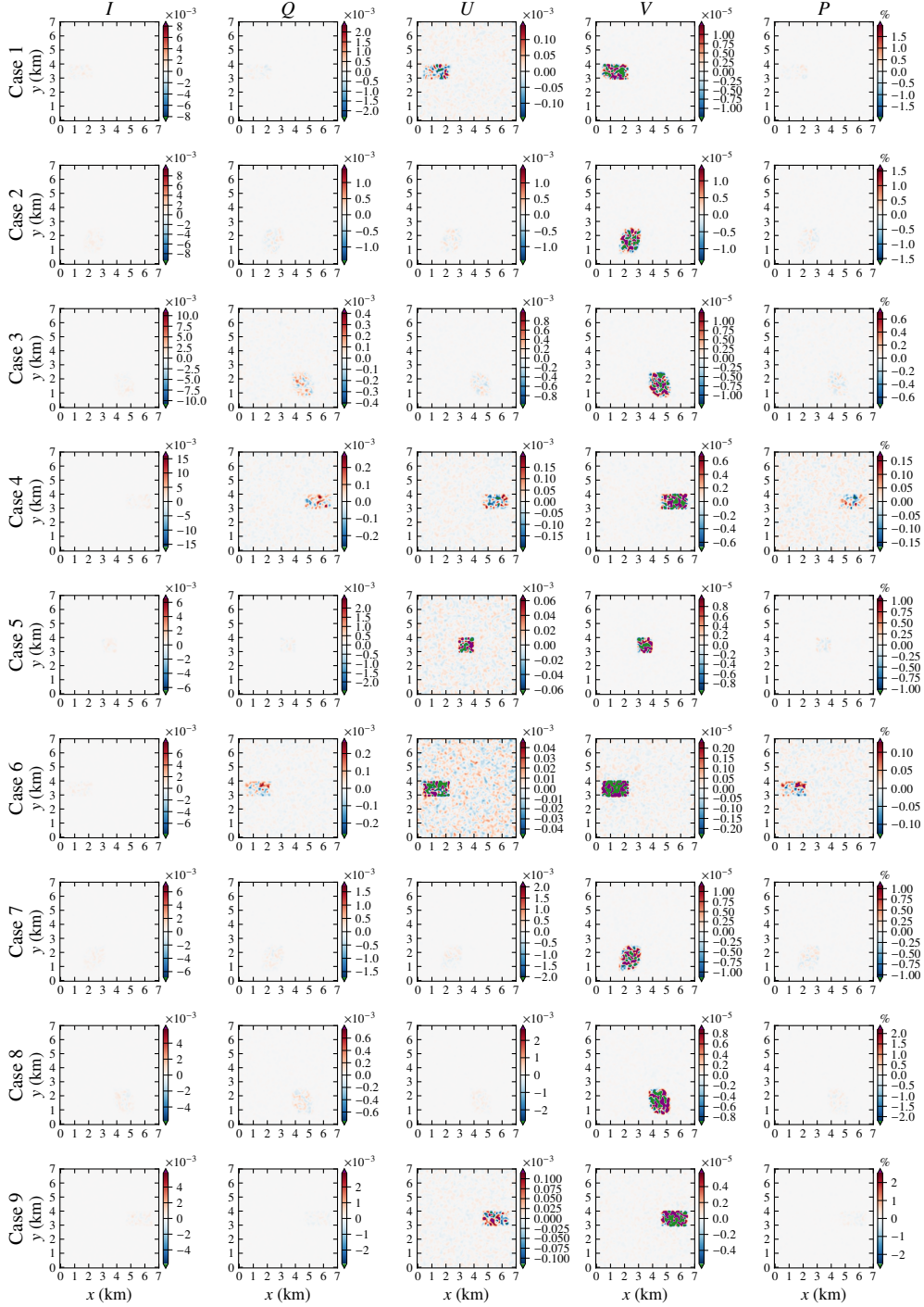


Figure S2. Similar to Fig. S1, except for the differences between MONKI and MYSTIC (see Fig. A2 of Emde et al., 2018). Following Emde et al. (2018), the limits of the colourbars are set to 5% of the maximum values for I and P , and to 10% of the values for $|Q|$, $|U|$ and $|V|$. Values larger than the upper limit are marked in purple and values smaller than the lower limit are marked in green.

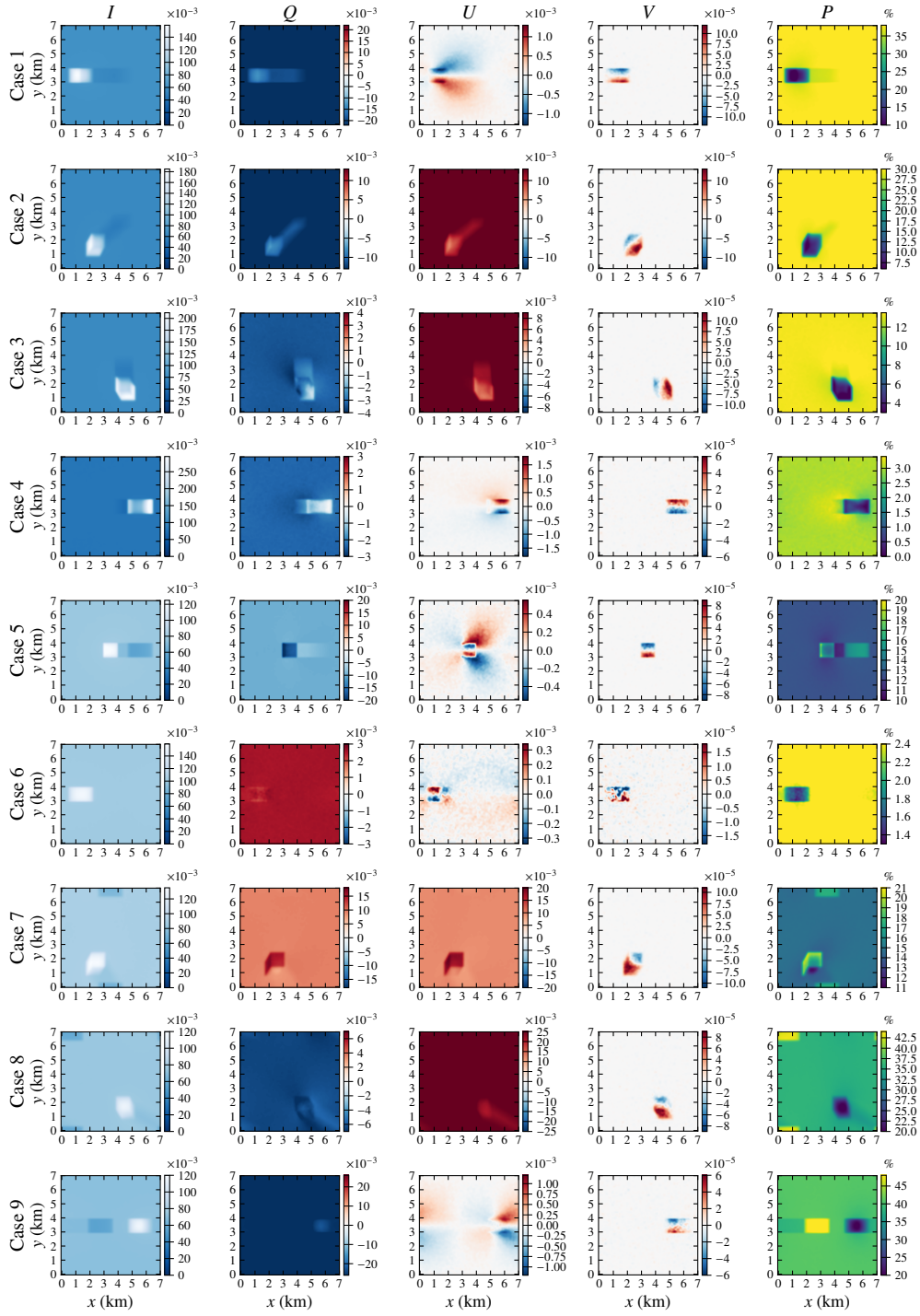


Figure S3. Similar to Fig. S1, except with MONKI in backward mode.

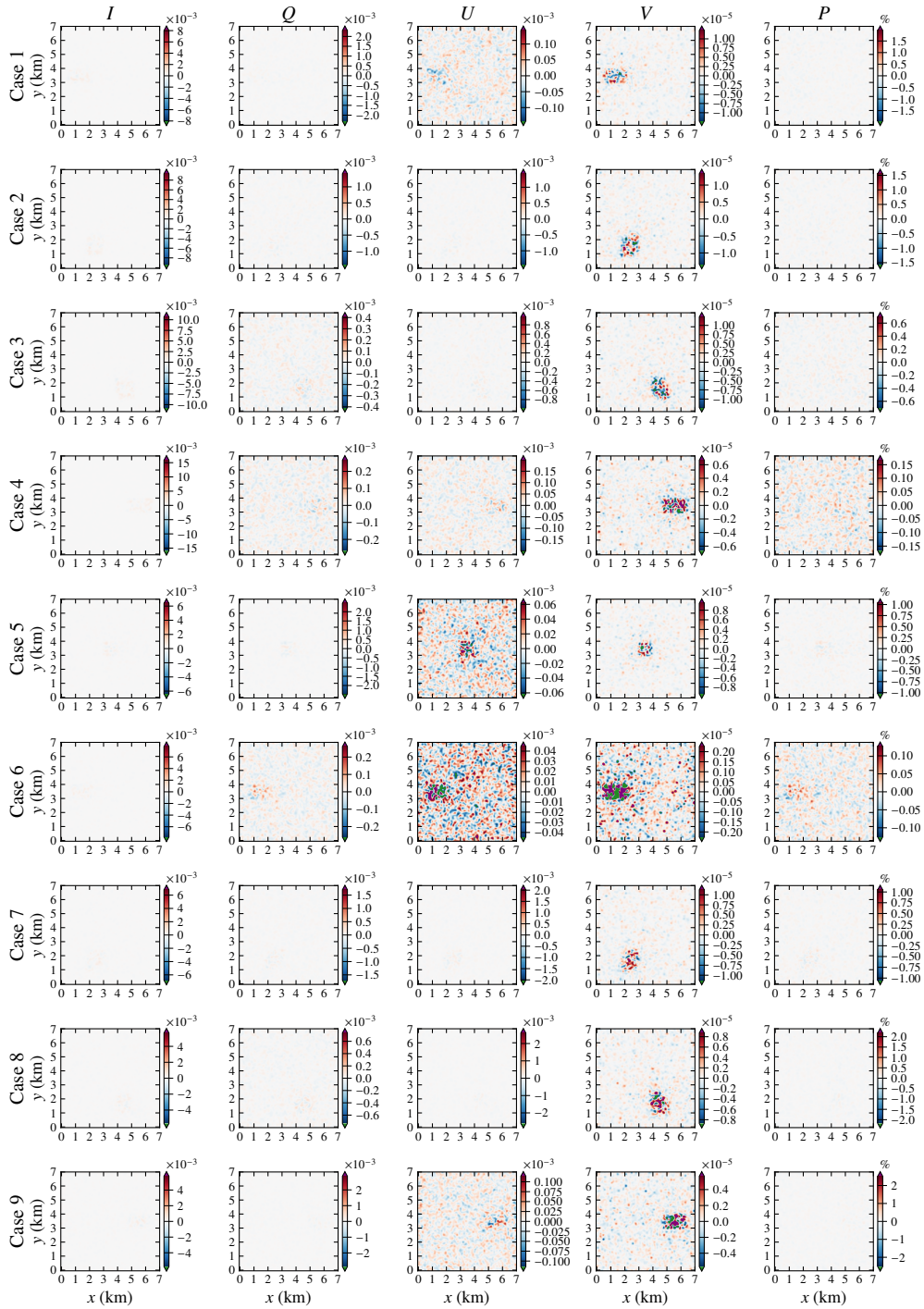


Figure S4. Similar to Fig. S2, except with MONKI in backward mode.

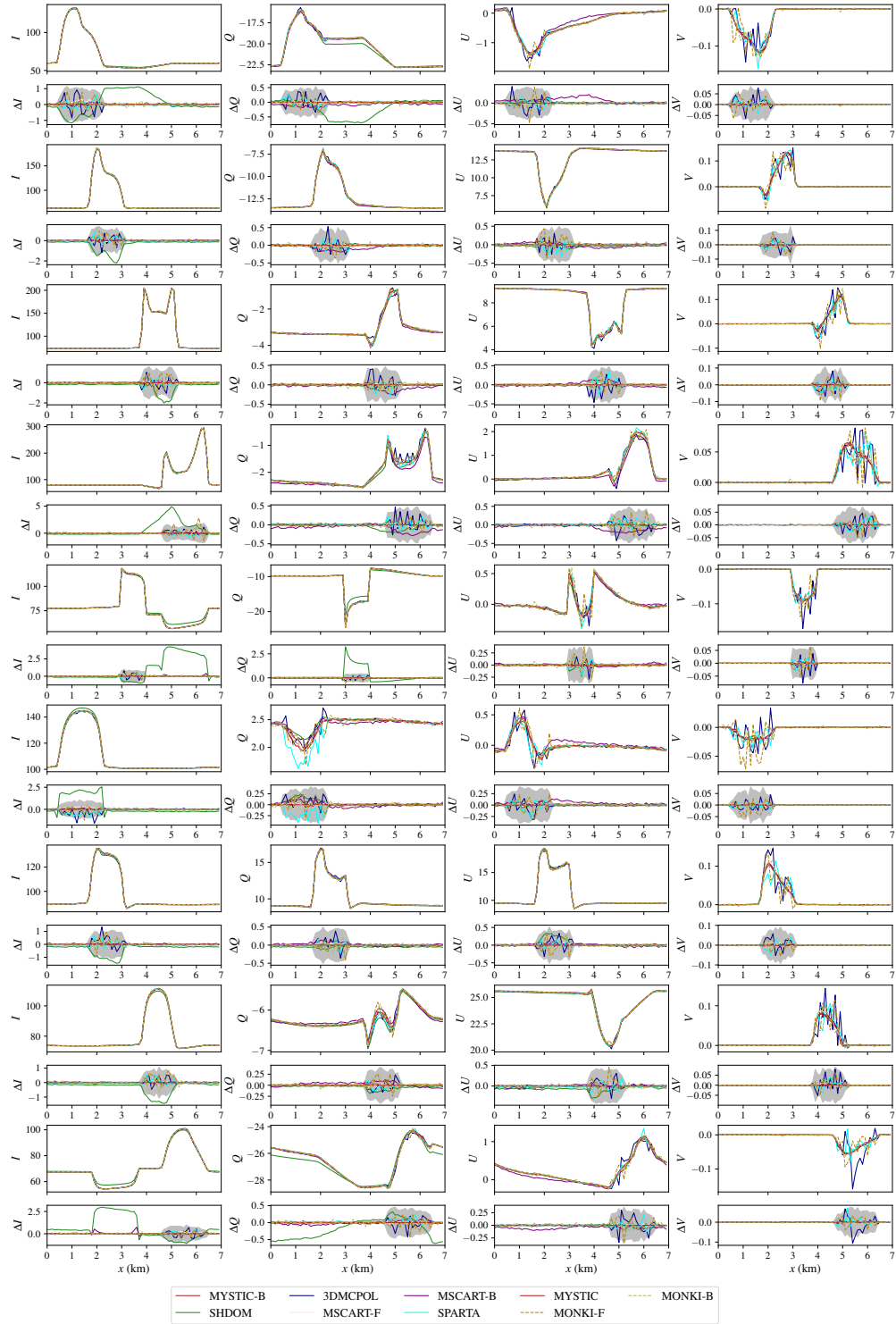


Figure S5. Similar to Fig. 8, except for the cases 1 to 9 (top to bottom row) with the geometries listed in Table S8.

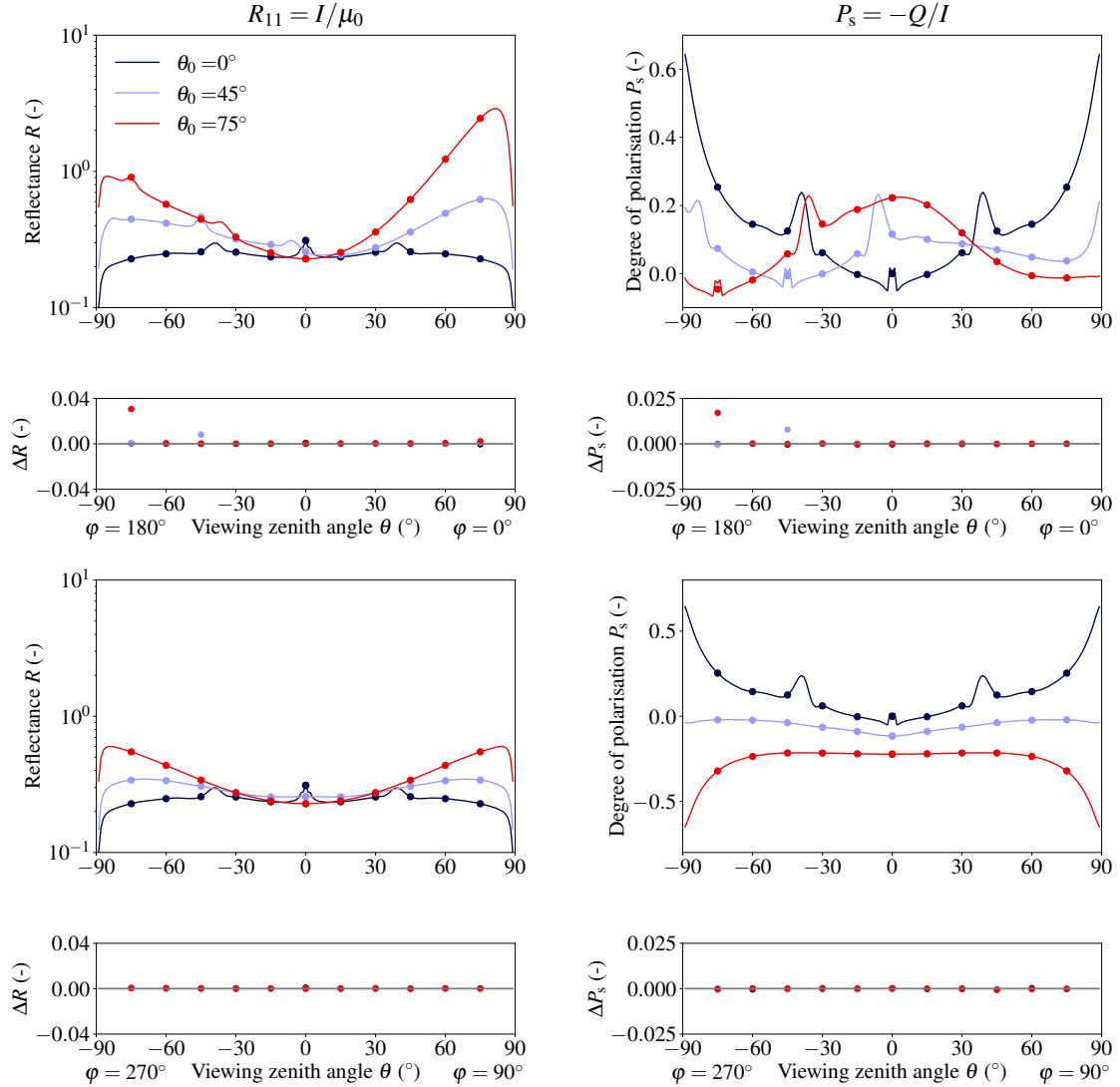


Figure S6. Comparison of the Earth model (see Sect. 4) results of MONKI (solid lines) and DAK (dots) for the TOA reflectance R_{11} and the degree of polarisation P_s , at the azimuth directions ($\varphi = 180^\circ$, $\varphi = 0^\circ$) (top row), and ($\varphi = 270^\circ$, $\varphi = 90^\circ$) (bottom row). The azimuth angle of the sunlight is $\varphi_0 = 0^\circ$. The differences (MONKI–DAK) are also shown. For this comparison, MONKI used 10^9 photons per (θ_0, φ) -combination and 90 viewing zenith angles θ simultaneously. In DAK, 170 Gaussian quadrature points were used for both μ and μ_0 , and 170 Fourier terms for the azimuth direction. This number of Gauss points in DAK explains the remaining differences between DAK and MONKI in the backscattering directions ($\theta_0 = 75^\circ$, $\theta = 75^\circ$, $\varphi = 180^\circ$) and ($\theta_0 = 45^\circ$, $\theta = 45^\circ$, $\varphi = 180^\circ$), where the sharp glory feature of the cloud droplets (at $\Theta \sim 180^\circ$ in Fig. 11) is not fully resolved by DAK’s Gaussian points.

List of symbols

Table S9. Radiometric and polarimetric quantities.

Symbol	Description	Unit
$\mathbf{I} = [I, Q, U, V]^T$	Radiance Stokes vector	$\text{W m}^{-2} \text{ nm}^{-1} \text{ sr}^{-1}$
I	Total radiance, or first Stokes component	$\text{W m}^{-2} \text{ nm}^{-1} \text{ sr}^{-1}$
Q	Linearly polarised radiance defined with respect to the chosen reference plane	$\text{W m}^{-2} \text{ nm}^{-1} \text{ sr}^{-1}$
U	Linearly polarised radiance at $\pm 45^\circ$ with respect to the chosen reference plane	$\text{W m}^{-2} \text{ nm}^{-1} \text{ sr}^{-1}$
V	Circularly polarised radiance	$\text{W m}^{-2} \text{ nm}^{-1} \text{ sr}^{-1}$
$\mathbf{I}' = [I', Q', U', V']^T$	Incident radiance Stokes vector at a scattering event	$\text{W m}^{-2} \text{ nm}^{-1} \text{ sr}^{-1}$
I'	First component of the incident radiance Stokes vector	$\text{W m}^{-2} \text{ nm}^{-1} \text{ sr}^{-1}$
$\mathbf{I}^\times$	Scattered Stokes vector after rotation to the scattering plane, used in the angular-PDF derivation	$\text{W m}^{-2} \text{ nm}^{-1} \text{ sr}^{-1}$
$I^\times$	First component of $\mathbf{I}^\times$	$\text{W m}^{-2} \text{ nm}^{-1} \text{ sr}^{-1}$
$\mathbf{I}(O, \boldsymbol{\omega}^\bullet; \boldsymbol{\omega}_0)$	Detector-area-averaged radiance Stokes vector for detector area O , detector direction $\boldsymbol{\omega}^\bullet$, and sunlight direction $\boldsymbol{\omega}_0$	$\text{W m}^{-2} \text{ nm}^{-1} \text{ sr}^{-1}$
$\mathbf{I}^{(0)}, \mathbf{I}^{(1)}, \mathbf{I}^{(2)}, \dots$	Zeroth-, first-, second-, and higher-order radiance Stokes vector contributions	$\text{W m}^{-2} \text{ nm}^{-1} \text{ sr}^{-1}$
$\mathbf{I}^{(n)}$	n^{th} -order radiance Stokes vector contribution	$\text{W m}^{-2} \text{ nm}^{-1} \text{ sr}^{-1}$
$\bar{\mathbf{I}}$	Normalised Stokes vector carried by a Monte Carlo photon	–
$\bar{\mathbf{I}}'$	Normalised incident Stokes vector in the angular-PDF derivation	–
$\bar{\mathbf{I}}_0$	Initial normalised Stokes vector of a photon in forward mode, equal to $\mathbf{1}$	–
$\bar{\mathbf{I}}_1, \bar{\mathbf{I}}_2, \dots$	Normalised radiance Stokes vectors after successive forward scattering events	–
E_0	Incident solar irradiance	$\text{W m}^{-2} \text{ nm}^{-1}$
$\mathbf{E}_0$	Incident solar flux Stokes vector	$\text{W m}^{-2} \text{ nm}^{-1}$
E	Reflected irradiance emerging from a Lambertian surface in Appendix C	$\text{W m}^{-2} \text{ nm}^{-1}$
$\mathbf{1}$	Unpolarised unit Stokes vector, $[1, 0, 0, 0]^T$	–
$\mathbf{e}_j$	Canonical basis vector in Stokes space	–
P	Degree of polarisation	–
P'	Degree of polarisation of the incident Stokes vector in the sampling derivations	–
P_s	Signed degree of linear polarisation, $-Q/I$	–
$\mathbf{R}$	Reflectance matrix of the local atmosphere-surface system	–
$\mathbf{T}$	Transmission matrix of the local atmosphere	–
R_{ij}	Element (i, j) of the reflectance matrix	–
R_{11}	Total TOA reflectance	–
R_{21}, R_{31}	Linearly polarised TOA reflectances	–
R_{41}	Circularly polarised TOA reflectance	–
$\mathbf{R}_L$	Lambertian surface-reflection matrix	–
$\mathcal{T}^{\text{bw}}$	Backward response quantity used in the derivation of the backward angular PDF	–

Table S10. Geometry, coordinates, and directions, part I.

Symbol	Description	Unit
x, y, z	Cartesian coordinates	m
$\mathbf{x} = [x, y, z]^T$	Position vector	m
$\mathbf{x}^\bullet$	Detector location or detector-boundary intersection point, depending on context	m
$\mathbf{x}^*$	Source point or dummy spatial integration variable in the Green-function derivation	m
$\mathbf{x}_0$	Entry point of the incident sunlight on the domain boundary; in the forward implementation also the photon injection point at the boundary	m
$\mathbf{x}_0(\mathbf{x})$	Entry point of the incident sunlight associated with spatial point $\mathbf{x}$	m
$\mathbf{x}_{0,i}$	Entry point of the incident sunlight associated with photon i in backward mode	m
$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$	Successive scattering locations along a photon path, labelled in forward order	m
$\mathbf{x}_{1,i}, \mathbf{x}_{2,i}, \dots$	Sampled scattering locations of photon i	m
$\mathbf{x}_{n,i,j}$	Location of event n of photon i injected in source column j	m
$\mathbf{x}_i^\bullet$	Detector-boundary intersection point associated with photon i	m
$\mathbf{x}_{n,i,j}^\bullet$	Intersection point of the detector ray associated with event (n, i, j) with the boundary at TOA or BOA	m
$\boldsymbol{\omega}$	Generic propagation-direction unit vector	–
$\boldsymbol{\omega}'$	Incident propagation-direction unit vector at a scattering event or in scattering integrals	–
$\boldsymbol{\omega}^\bullet$	Detector/viewing direction	–
$\boldsymbol{\omega}_0$	Solar-beam direction	–
$\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \dots$	Intermediate photon directions between successive scattering events	–
$\boldsymbol{\omega}_{n,i}$	Photon direction associated with event or order n of photon i	–
$\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \dots$	Randomly sampled propagation directions between successive scattering events	–
$\hat{\mathbf{r}}$	Unit vector from a source point towards the observation point in the Green-function derivation	–
$\ \mathbf{x} - \mathbf{x}^*\ $	Distance between the observation point $\mathbf{x}$ and source point $\mathbf{x}^*$	m

Table S11. Geometry, coordinates, and directions, part II.

Symbol	Description	Unit
ϑ, φ	Zenith and azimuth angles of a generic propagation direction	rad or $^\circ$
ϑ', φ'	Zenith and azimuth angles of the incident direction at a scattering event	rad or $^\circ$
ϑ_0, φ_0	Zenith and azimuth angles of the incident solar beam	rad or $^\circ$
$\vartheta^\bullet, \varphi^\bullet$	Zenith and azimuth angles of the detector direction	rad or $^\circ$
ϑ_n, φ_n	Zenith and azimuth angles of the photon direction after scattering event n	rad or $^\circ$
θ	Viewing zenith angle, $\arccos(\mu)$, defined between 0 and $\pi/2$ with respect to the local normal	rad or $^\circ$
θ_0	Solar zenith angle, $\arccos(\mu_0)$, defined between 0 and $\pi/2$ with respect to the local normal	rad or $^\circ$
μ	$ \cos \vartheta $	–
μ_0	$ \cos \vartheta_0 $	–
$\mu_\bullet$	$ \cos \vartheta^\bullet $	–
Θ	Scattering angle	rad or $^\circ$
σ_1, σ_2	Rotation angles between the meridian planes and the scattering plane	rad or $^\circ$
Ω	Solid angle	sr
$d\Omega$	Differential solid-angle element	sr
$d\Omega'$	Differential solid-angle element for the incident direction in scattering integrals	sr
O_0	Horizontal source area on the illuminated sunlight-entry boundary in the finite-area forward formulation	m^2
O	Horizontal detector area or grid-column area, depending on context	m^2
dO, dO_0	Differential horizontal detector and source area elements	m^2
dV	Differential scattering volume element	m^3
$d^3\mathbf{x}$	Differential volume element	m^3
$\mathcal{V}$	Spatial domain occupied by the atmosphere	m^3

Table S12. Grid, indices, and model-domain quantities.

Symbol	Description	Unit
N_x, N_y, N_z	Numbers of grid columns in the x and y directions and layers in the z direction	–
i	Photon index; also used as absorbing-gas species index in Appendix A, depending on context	–
j	Source-column index in the multi-column estimator; also used as Stokes-component index in the derivation, depending on context	–
k	Vertical layer index	–
k^*	Index of the layer containing a given event	–
l	Dummy product index over scattering events	–
m	Backward detector-side event counter in the detector-response matrix recursion	–
n	Scattering order or event index	–
r	Simulation-round index	–
c	Aerosol or cloud-particle type index	–
t, t'	Atmospheric constituent type indices, referring to gas molecules or aerosol types	–
z_k^{bot}	Bottom altitude of layer k	m or km
z_k^{top}	Top altitude of layer k	m or km
Δz_k	Thickness of layer k	m or km

Table S13. Optical coefficients, matrices, and surface quantities.

Symbol	Description	Unit
λ	Wavelength	nm or m
β	Total extinction coefficient	m^{-1}
β_{sca}	Total scattering coefficient	m^{-1}
β_{abs}	Total absorption coefficient	m^{-1}
$\beta_t, \beta_{t'}$	Total extinction coefficient of constituent type t or t'	m^{-1}
$\beta_{\text{sca},t}$	Scattering coefficient of constituent type t	m^{-1}
β^{m}	Gas extinction coefficient	m^{-1}
$\beta_{\text{sca}}^{\text{m}}, \beta_{\text{abs}}^{\text{m}}$	Gas scattering and absorption coefficients	m^{-1}
β_c^{a}	Extinction coefficient of aerosol type c	m^{-1}
$\beta_{\text{sca},c}^{\text{a}}, \beta_{\text{abs},c}^{\text{a}}$	Scattering and absorption coefficients of aerosol type c	m^{-1}
β_k	Extinction coefficient of layer k in the precomputation of optical paths	m^{-1}
$\beta_{n,i,j}(s')$	Extinction coefficient encountered along the event-to-detector path associated with event (n, i, j)	m^{-1}
a	Single-scattering albedo of the local mixture	–
a_t	Single-scattering albedo of selected constituent type t	–
a^{m}	Single-scattering albedo of the gas	–
a_c^{a}	Single-scattering albedo of aerosol type c	–
b	Total optical thickness	–
b_t	Optical thickness of constituent type t	–
$b_{\text{sca},t}, b_{\text{abs},t}$	Scattering and absorption optical thicknesses of constituent type t	–
b^{m}	Gas optical thickness	–
b^{a}	Aerosol or cloud optical thickness	–
$b_{\text{sca}}^{\text{m}}, b_{\text{abs}}^{\text{m}}$	Gas scattering and absorption optical thicknesses	–
$b_{\text{sca},c}^{\text{a}}, b_{\text{abs},c}^{\text{a}}$	Scattering and absorption optical thicknesses of aerosol type c	–
$\tau(\mathbf{x}_a \rightarrow \mathbf{x}_b)$	Optical depth between two points $\mathbf{x}_a$ and $\mathbf{x}_b$	–
T_{dir}	Direct transmittance	–
$\mathbf{F}$	Effective single-scattering matrix of the local atmospheric mixture	–
$\mathbf{F}_t$	Single-scattering matrix of selected constituent type t	–
$\mathbf{F}^{\text{m}}$	Rayleigh single-scattering matrix of the gas	–
$\mathbf{F}_c^{\text{a}}$	Mie single-scattering matrix of aerosol type c	–
F_{ij}	Element (i, j) of a single-scattering matrix	–
F_{11}	Phase function, i.e. the $(1, 1)$ element of the single-scattering matrix	–
$\mathbf{Z}$	Phase matrix including reference-plane rotations	–
$\mathbf{Z}_t$	Phase matrix of selected constituent type t	–
$\mathbf{L}$	Rotation matrix for Stokes vectors	–
α	Rotation angle in $\mathbf{L}(\alpha)$	rad or $^\circ$
A_{s}	Lambertian surface albedo	–
Δ_3	Diagonal matrix $\text{diag}(1, 1, -1, 1)$ used in the reciprocity relation	–
$\text{Pr}(t \mid \mathbf{x}, \lambda)$	Probability that constituent type t is selected at position $\mathbf{x}$ and wavelength λ	–
$\text{Pr}(d \leq s)$	Probability that the free photon path length d is smaller than or equal to distance s	–

Table S14. Atmospheric and particle properties.

Symbol	Description	Unit
ρ	Depolarisation factor	–
Δ, Δ'	Rayleigh-scattering coefficients derived from ρ	–
F_K	King correction factor	–
n	Refractive index of the gas	–
n_r, n_i	Real and imaginary parts of the particle refractive index	–
$\sigma_{\text{abs},i}^{\text{m}}$	Absorption cross section of gas species i	m^2
σ_c^{a}	Extinction cross section of aerosol type c	m^2
η_i	Volume mixing ratio of gas species i	–
N_g	Number of absorbing gas species	–
N_c	Number of aerosol or cloud-particle types	–
$N_{\text{col},c}^{\text{a}}$	Particle column number density of aerosol type c	m^{-2}
r_{eff}	Effective particle radius	μm
v_{eff}	Effective variance of the particle size distribution	–
$p(z)$	Pressure profile	Pa
$T(z)$	Temperature profile	K
g	Gravitational acceleration	m s^{-2}
M	Molar mass of the background gas	kg mol^{-1}
R	Universal gas constant	$\text{J mol}^{-1} \text{K}^{-1}$
N_{av}	Avogadro constant	mol^{-1}
N_L	Loschmidt number	m^{-3}
Δp_k	Pressure difference across layer k	Pa

Table S15. Monte Carlo path, weight, and estimator quantities, part I.

Symbol	Description	Unit
N	Number of photons launched per source column in forward mode or per detector column in backward mode	–
N_r	Number of independent simulation rounds	–
M_i	Total number of scattering events of photon i	–
$M_{i,j}$	Total number of events of photon i launched in source column j	–
w	Photon weight	–
$w_{n,i}$	Photon weight at scattering order or event n of photon i	–
$w_{n,i,j}$	Photon weight at event n of photon i launched in source column j	–
$\mathbf{X}_0$	Random forward launch position on the source area O_0	m
$\mathbf{X}^\bullet$	Random backward launch position on the detector area O	m
d	Free photon path length to the next atmospheric event	m
Δs	Distance from an event location to the nearest layer boundary along the detector path	m
s	Generic line-of-sight distance variable	m
s'	Dummy path-integration variable	m
s^*	Upstream distance variable in the Green-function derivation	m
s_b	Distance from the observation point to the upstream boundary along a ray	m
$s^\bullet$	Distance from an event location to the detector boundary	m
$s_{n,i,j}^\bullet$	Distance from event (n, i, j) to the detector boundary along the detector path	m
s_0	Solar-side distance from a scattering point to the sunlight-entry boundary along $-\omega_0$; also used as a free-path variable in the forward derivation	m
$s_{0,i}$	Solar-side distance from $\mathbf{x}_{1,i}$ to the sunlight-entry boundary in the backward estimators	m
s_1, s_2, s_3	Path-length variables between successive scattering events in the deterministic derivations	m
S_0, S_1, S_2, S_3	Random free-path lengths in forward Monte Carlo	m
$S_{0,i}, S_{1,i}, S_{2,i}, S_{3,i}$	Realisations of sampled forward free-path lengths for photon i	m
ξ, ξ_1, ξ_2, ξ_3	Independent random numbers uniformly distributed on $(0, 1)$	–

Table S16. Monte Carlo path, weight, and estimator quantities, part II.

Symbol	Description	Unit
$\mathbf{W}$	Local-estimation contribution to the detector signal in forward mode	–
$\mathbf{W}_{n,i,j}$	Forward local-estimation contribution of event n of photon i launched in source column j	–
$\mathbf{W}^{\text{bw}}$	Local-estimation contribution to the detector signal in backward mode	–
$\mathbf{W}_{n,i}^{\text{bw}}$	Backward local-estimation contribution of event n of photon i	–
χ	Indicator function selecting detector rays that intersect detector area O	–
$\chi_{n,i,j}$	Indicator for event (n, i, j) in the practical forward multi-column estimator	–
$\overline{\mathbf{M}}$	Normalised detector-response matrix in backward Monte Carlo	–
$\overline{\mathbf{M}}'$	Updated normalised detector-response matrix after a backward-mode event	–
$\overline{\mathbf{M}}_0, \overline{\mathbf{M}}_1, \overline{\mathbf{M}}_2, \dots$	Updated detector-response matrices after successive backward events	–
$\overline{M}_{12}, \overline{M}_{13}$	Elements (1, 2) and (1, 3) of $\overline{\mathbf{M}}$	–
$[\cdot]_1$	Operation selecting the first component of a Stokes vector	–
I_r	Simulated first Stokes parameter in simulation round r	$\text{W m}^{-2} \text{ nm}^{-1} \text{ sr}^{-1}$
$\langle I \rangle$	Mean value of I over simulation rounds	$\text{W m}^{-2} \text{ nm}^{-1} \text{ sr}^{-1}$
$\sigma_{\langle I \rangle}$	Standard error of the mean of I	$\text{W m}^{-2} \text{ nm}^{-1} \text{ sr}^{-1}$

Table S17. Probability densities, cumulative distributions, and sampling functions.

Symbol	Description	Unit
$p(\cdot)$	Probability density function, generic notation, including joint PDFs	depends on argument
$P(\cdot)$	Cumulative distribution function in the sampling appendices	–
p_{O_0}	PDF of the forward launch position on the source area O_0	m^{-2}
p_O	PDF of the backward launch position on the detector area O	m^{-2}
$p_{S_0}, p_{S_1}, p_{S_2}, p_{S_3}$	PDFs of sampled free paths in forward Monte Carlo	m^{-1}
$p_{S_0}^{\text{bw}}, p_{S_1}^{\text{bw}}, p_{S_2}^{\text{bw}}, p_{S_3}^{\text{bw}}$	PDFs of sampled free paths in backward Monte Carlo; the index depends on the scattering order considered	m^{-1}
$p_{\Omega_1}, p_{\Omega_2}$	PDFs of sampled scattering directions in forward Monte Carlo	sr^{-1}
$p_{\Omega_1}^{\text{bw}}, p_{\Omega_2}^{\text{bw}}$	PDFs of sampled scattering directions in backward Monte Carlo	sr^{-1}
$p(\mu, \varphi)$	Joint PDF of the Lambertian reflection direction	–
$p(\mu)$	Marginal PDF of μ for Lambertian reflection	–
$p(\varphi)$	Marginal PDF of φ for Lambertian reflection	–
$p(\cos \Theta, \sigma_1)$	Joint PDF of the forward scattering direction of polarised light	–
$p^{\text{bw}}(\cos \Theta, \sigma_2)$	Joint PDF of the backward scattering direction	–
f	Envelope function in acceptance-rejection sampling	same as target PDF
g	Proposal PDF in acceptance-rejection sampling	same as target PDF
C	Normalisation or rejection constant in acceptance-rejection sampling	–
G	CDF of the proposal distribution in the Mie-sampling appendix	–
G^{-1}	Inverse CDF used in inverse-transform sampling, particularly for Mie scattering	depends on argument
X	Auxiliary quantity in the forward sampling derivations	–
Y	Auxiliary quantity in the backward sampling derivations	–

Table S18. Vector radiative transfer equation, collision-density, and operator symbols.

Symbol	Description	Unit
∇	Spatial gradient operator	m^{-1}
$\mathcal{L}$	Linear transport operator in the VRTE derivation	–
$\mathcal{L}_s$	One-dimensional transport operator along a ray	–
$\mathcal{S}_0$	Source-density term in the VRTE	$\text{W m}^{-3} \text{ nm}^{-1} \text{ sr}^{-1}$
q	Local source term in the formal VRTE solution	$\text{W m}^{-2} \text{ nm}^{-1} \text{ sr}^{-1}$
f	Collision-density Stokes vector	$\text{W m}^{-3} \text{ nm}^{-1} \text{ sr}^{-1}$
Ψ	Initial collision density	$\text{W m}^{-3} \text{ nm}^{-1} \text{ sr}^{-1}$
k	Transport kernel in the collision-density formulation	$\text{m}^{-3} \text{ sr}^{-1}$
K	Integral transport operator acting on the collision density	–
K^+	Adjoint transport operator	–
$(\mathbf{u}, \mathbf{v})$	Inner product of Stokes-vector-valued functions	depends on arguments
$\mathbf{u}, \mathbf{v}, \mathbf{g}$	Generic Stokes-vector-valued functions used in the operator derivations	depends on context
$\delta(\cdot)$	Dirac delta function	inverse of argument unit
$\delta_{\mathbf{x}} \bullet \boldsymbol{\omega} \bullet$	Product Dirac delta function selecting detector point and propagation direction	$\text{m}^{-3} \text{ sr}^{-1}$
Φ_j	Contribution function for the j^{th} detector Stokes component	$\text{m}^{-2} \text{ sr}^{-1}$

Table S19. Scattering-order, finite-area, and parameterised-path symbols.

Symbol	Description	Unit
J	Generic vector-valued integral used in the Monte Carlo-estimator derivations	depends on context
g_1, g_2, g_3	Integrands used in the first-, second-, and third-order forward Monte Carlo-estimator derivations	depends on context
$g_1^{\text{bw}}, g_2^{\text{bw}}, g_3^{\text{bw}}$	Integrands used in the first-, second-, and third-order backward Monte Carlo-estimator derivations	depends on context
$h(\mathbf{x}_1)$	Generic integrable function of the scattering point used in the finite-area transformation	depends on context
$s_0(\mathbf{x})$	Solar-side distance from $\mathbf{x}$ to the sunlight-entry boundary along $-\boldsymbol{\omega}_0$	m
$\mathbf{x}_1(s_1)$	First-scattering location along the detector line of sight in the first-order derivation	m
$\mathbf{x}_0(s_1)$	Entry point of the incident sunlight associated with $\mathbf{x}_1(s_1)$	m
$\mathbf{x}_2(s_2)$	Second, or last, scattering location along the detector line of sight in the second-order derivation	m
$\mathbf{x}_1(s_2, s_1, \boldsymbol{\omega}_1)$	First-scattering location in the second-order derivation	m
$\mathbf{x}_0(s_2, s_1, \boldsymbol{\omega}_1)$	Entry point of the incident sunlight associated with $\mathbf{x}_1(s_2, s_1, \boldsymbol{\omega}_1)$	m
$\mathbf{x}_3(s_3)$	Third, or last, scattering location along the detector line of sight in the third-order derivation	m
$\mathbf{x}_2(s_3, s_2, \boldsymbol{\omega}_2)$	Second-scattering location in the third-order derivation	m
$\mathbf{x}_1(s_3, s_2, s_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_1)$	First-scattering location in the third-order derivation	m
$\mathbf{x}_0(s_3, s_2, s_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_1)$	Entry point of the incident sunlight associated with $\mathbf{x}_1(s_3, s_2, s_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_1)$	m

Table S20. Classical sampling-approach symbols.

Symbol	Description	Unit
$p_{\Omega_2}^{(0)}$	Approximate forward PDF for sampling ω_2 when the incident beam is treated as unpolarised	sr^{-1}
$p_{\Omega_1}^{\text{bw},(0)}$	Approximate backward PDF for sampling ω_1 when the incident beam is treated as unpolarised	sr^{-1}
C_{Ω_2}	Importance-sampling correction factor in the classical forward approach	–
$C_{\Omega_1}^{\text{bw}}$	Importance-sampling correction factor in the classical backward approach	–

Table S21. Symbols used in the Green-function discussion.

Symbol	Description	Unit
$u(x)$	Unknown scalar function in the generic boundary-value problem	–
$q(x)$	Source term in the generic boundary-value problem	–
x	One-dimensional coordinate in the generic Green-function example	–
x^*	Source location in the one-dimensional Green-function example	–
a, b	Interval endpoints in the generic boundary-value problem	–
B	Interval length or upper boundary in the worked Green-function example	–
α	Positive constant in the worked Green-function example	depends on context
$G(x, x^*)$	Green's function in the generic one-dimensional example	–
$G(s, s^*)$	Green's function along a ray in the VRTE derivation	m^{-1}
H	Heaviside step function	–
$C_1(x^*), C_2(x^*)$	Integration constants in the worked Green-function example	–
$C_-(s^*), C_+(s^*)$	Integration constants for the Green's function on either side of the source point s^*	depends on context
ϵ	Infinitesimal interval half-width used to derive the Green-function jump condition	same unit as s
s^{*-}, s^{*+}	Points immediately downstream and upstream of the source point s^* in the Green-function derivation	m