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Supplement of

Love number computation within the Ice-sheet and Sea-level System Model (ISSM v4.24)

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S1: Alternative analytical form of hypergeometric functions for EBM rheology

This supplement describes a simple procedure to be implemented for the hypergeometric function, $\Upsilon =_2 F_1[1,1+\alpha;2+\alpha;z]$; $z=-s\tau_{H,L}$ for the extended Burgers Material model with $0<\alpha<1$ without requiring a function call to software for the general case of the hypergeometric function. The main purpose of avoiding such a function call is to gain some computational efficiency by tailoring the replacement to the detailed requirements of the EBM/GIA ensemble for Bayesian analysis. While past experience suggests that the α-dependency is weak, further testing is required. A method relied upon here is to employ Mathematica software.

For example, Mathematica returns the specific case of $\alpha = \frac{1}{2}$:

$$ {}_2F_1[1,\frac{3}{2};\frac{5}{2};z] \;=\; \frac{3(-z+\sqrt{2}\text{ArcTan}\sqrt{z})}{z^2}. \tag{S1}$$

We remind the reader that all actions on the variable, z, for transcendental equations assume complex computation, and similarly for the hypergeometric function. There are no straightforward simplifications for ${}_2F_1$ offered by Mathematica 13.1, or earlier versions. Furthermore no useful reductions may be derived from Abramowitz and Stegun (1970) in any obvious way. The functions ${}_2F_1[a,b;c;z]$ may be recovered from the following definition

$$ {}_2F_1[a,b;c;z] \;\equiv\; \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{p=0}^{\infty}\frac{\Gamma(a+p)\Gamma(b+p)}{\Gamma(c+p)}\,\frac{z^p}{p!}. \tag{S2}$$

where Γ represents the Gamma function, p is an integer and the mark ! indicating factorial. However, the functions are also a special solution of the equation:

$$ z(1-z)\frac{d^2\Upsilon}{d^2z}+[c-(a+b+1)z]\frac{d\Upsilon}{dz}-ab\Upsilon \;=\; 0. \tag{S3}$$

This equation is sometimes called Gauss' differential equation. Solutions have three regular points [e.g.,Wylie 1966, p. 653, Mathews and Walker 1970, p. 187]; $z=0,1,\infty$. The solution ${}_2F_1[a,b;c;z]$ behaves as a constant near $z=0$. If this constant is equal to unity, we formally arrive at the hypergeometric function ${}_2F_1$.

We may treat α as a ratio of two integers, m and n where $m<n$, and both n and m are positive. In our case $a=1$, $b=(n+m)/n$ and $c=(2n+m)/n$. There is no advantage to this substitution other than it provides a way to solve special cases of the one point boundary value problem

$$ nz(1-z)\frac{d^2\Upsilon}{d^2z}+[2n+m-(3n+m)z]\frac{d\Upsilon}{dz}-(n+m)\Upsilon \;=\; 0, \tag{S4}$$

with the condition

$$ \Upsilon(0) \;=\; 1. \tag{S5}$$

For example, when $m=1, n=4$ ($\alpha=\frac{1}{4}$), then the solution for $\Upsilon(z)$ is

$$ \Upsilon(z) \;=\; -\frac{5}{2z^{\frac{5}{4}}}\cdot\{2z^{\frac{1}{4}}-\text{ArcTan}[z^{\frac{1}{4}}]-\text{ArcTanh}[z^{\frac{1}{4}}]\}. \tag{S6}$$

Possibly a more illustrative example is when $m = 1, n = 3$ ($\alpha = \frac{1}{3}$). A more complicated expression is generated. The solution is

$$\Upsilon(z) = -\frac{2}{9z^{\frac{4}{3}}} \cdot \{(\sqrt{3} - 6i)\pi + 3(6z^{\frac{1}{3}} - 2\sqrt{3}ArcTan[\frac{2z^{\frac{1}{3}}+1}{\sqrt{3}}] + 2ln[z^{\frac{1}{3}} - 1] - ln[z^{\frac{1}{3}} + z^{\frac{2}{3}} + 1])\} \tag{S7}$$

where i is the imaginary unit $\sqrt{-1}$. Clearly, complex arithmetic should be fully implemented when using the simplifications owing to a Gauss' differential equation solution method for generating the special cases that arise in the power law distribution function of EBM theory. Figure S1 shows a comparison of the two methods, one which we term 'exact' is obtained in a function call in Mathematica computing language and the Gauss' differential equation method for the case $\alpha = \frac{1}{3}$. Some of the expressions generated by this method are more complicated than in our two examples, but these will always be more efficient than general call to functions generating ${}_2F_1[1, 1+\alpha; 2+\alpha; -\tau_j s/\tau_M]$. This is especially important due to the necessity of inverting the Laplace transforms numerically in order to generate time-dependent GIA solutions.

While transcendental equations and hypergeometric functions assume complex argument, computation of the imaginary parts reveal they are numerically near zero. EBM cases for $\alpha = \frac{1}{3}$, for example, reveal the following results using the Gauss method;

$$\Upsilon(-10001.0) \;=\; 0.000377533 \;+\; 1.6263 \times 10^{-19} i, \tag{S8}$$

$$\Upsilon(-.0001) \;=\; 0.999943 \;-\; 2.63911 \times 10^{-11} i, \tag{S9}$$

with the default precision in Mathematica on a MacBook Pro with Apple M1 Max chip.

In addition to the cases of α at values $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, as discussed here, we also give three additional solutions using the Gauss differential equation method. Case $\alpha = \frac{1}{8}$ has the solution

$$\begin{aligned}\Upsilon(z) \;&=\; -\frac{9}{z} + \frac{9}{16z^{\frac{9}{8}}}\{\sqrt{2}(\pi + 2\,\mathrm{ArcCoth}[\frac{1+z^{\frac{1}{4}}}{\sqrt{2}z^{\frac{1}{8}}}]) \\ &\quad + 4\,(\mathrm{ArcTan}[z^{\frac{1}{8}}] + \mathrm{ArcTanh}[z^{\frac{1}{8}}]) + 2\sqrt{2}\,\mathrm{ArcTanh}[\frac{\sqrt{2}z^{\frac{1}{8}}}{1+z^{\frac{1}{8}}}]\}.\end{aligned} \tag{S10}$$

Case $\alpha = \frac{2}{3}$ has the solution

$$\begin{aligned}\Upsilon(z) \;&=\; -\frac{1}{18z^{\frac{5}{3}}} \cdot \{(\sqrt{3} - 6i)\pi + 9z^{\frac{2}{3}} + \mathrm{ArcTan}[\frac{1+2z^{\frac{1}{3}}}{\sqrt{3}}] \\ &\quad + 3\,(2\ln[-1 + z^{\frac{1}{3}}] - \ln[1 + z^{\frac{1}{3}} + z^{\frac{2}{3}})])\},\end{aligned} \tag{S11}$$

and case $\alpha = \frac{3}{4}$, the solution;

$$\Upsilon(z) \;=\; -\frac{7}{6z^{\frac{7}{4}}} \cdot \{2z^{\frac{3}{4}} + 3(\mathrm{ArcTan}[z^{\frac{1}{4}}] - \mathrm{ArcTanh}[z^{\frac{1}{4}}])\}. \tag{S12}$$

It is also possible to pursue an examination of the case of $\alpha = 0$ by taking the limit of the α-dependent part of Equation (19) of Ivins et al [2022]. This leads to a form that contains the sum on the low and high cutoff times (τ_L and τ_H) of $Ei[-t/\tau_j]$

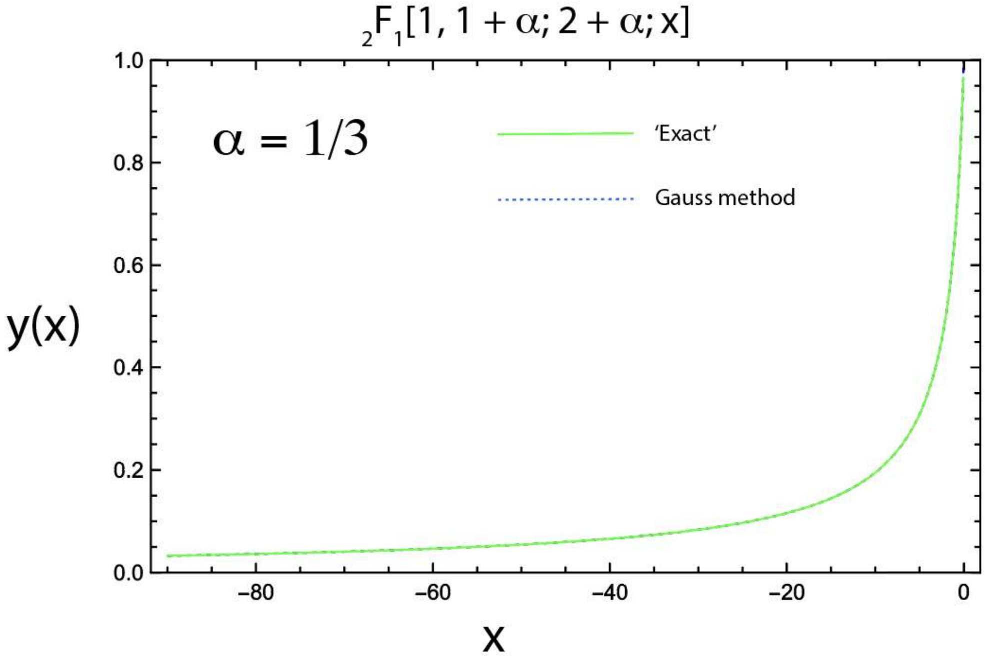

Figure S1. A comparison of the hypergeometric function computation options required of EBM/GIA.

with sum on the two cutoff times, τ_j. To complete this solution for α = 0, then requires convolving the load function with this exponential integral function, $Ei[-t/\tau_j]$, a task we can leave to the reader. It is noted that laboratory torsion experimental values of α are all larger than $\frac{1}{8}$.

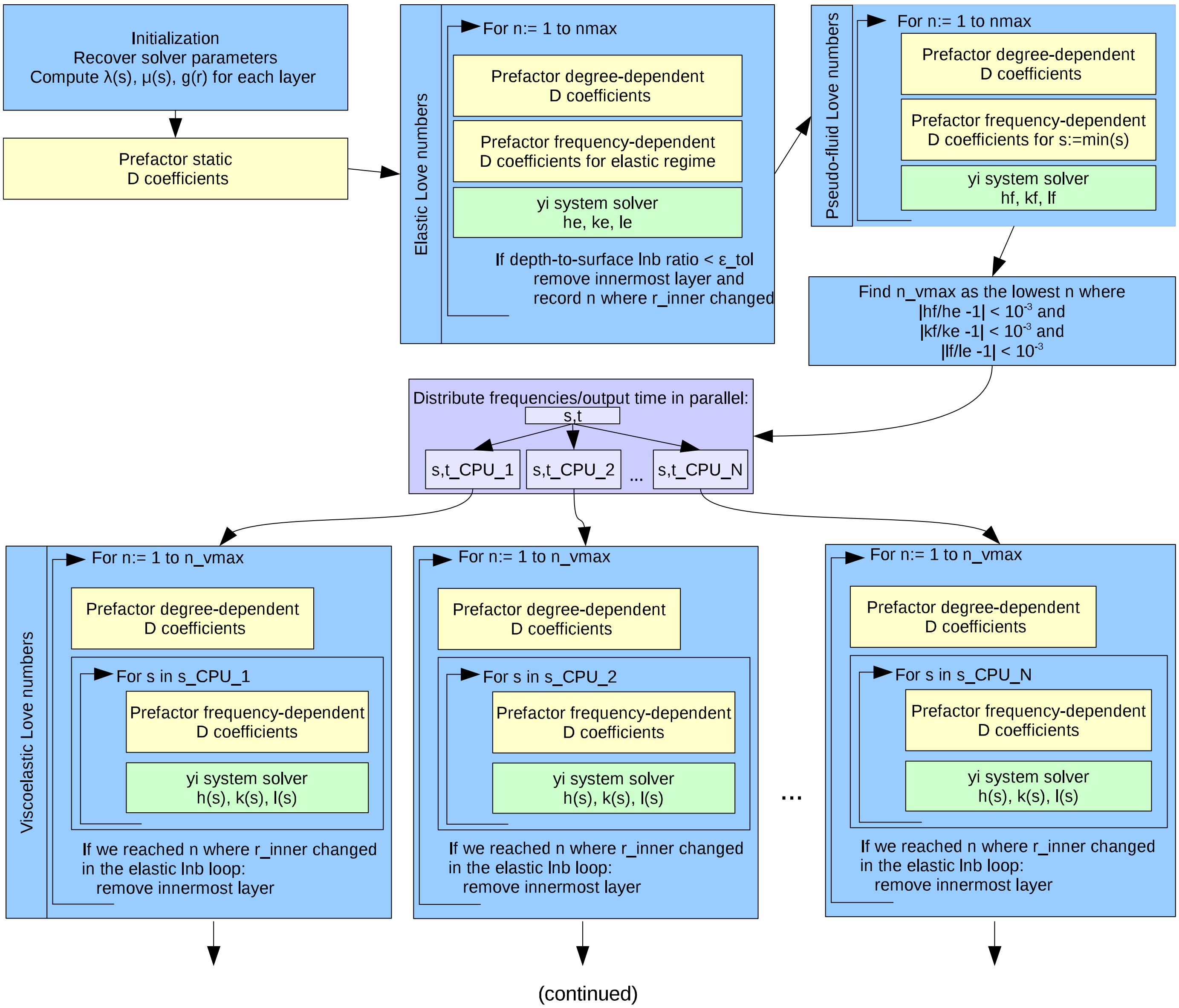

Figure S2. Flowchart of the Love number solver algorithm in its fully optimized form. Yellow boxes indicate optimization steps where prefactoring of matrix $\bar{\bar{\mathbf{D}}}$ coefficients is computed. "Lnb" is short for Love numbers. Green boxes indicate that the resolution of the y_i system is performed (see fig. S3). Boxes in dark blue indicate where the Post-Widder algoritm for inverse Laplace Transform is applied. Purple boxes indicate steps where communication between CPUs in the parallelization process is occurring. All other operations are indicated by blue boxes.

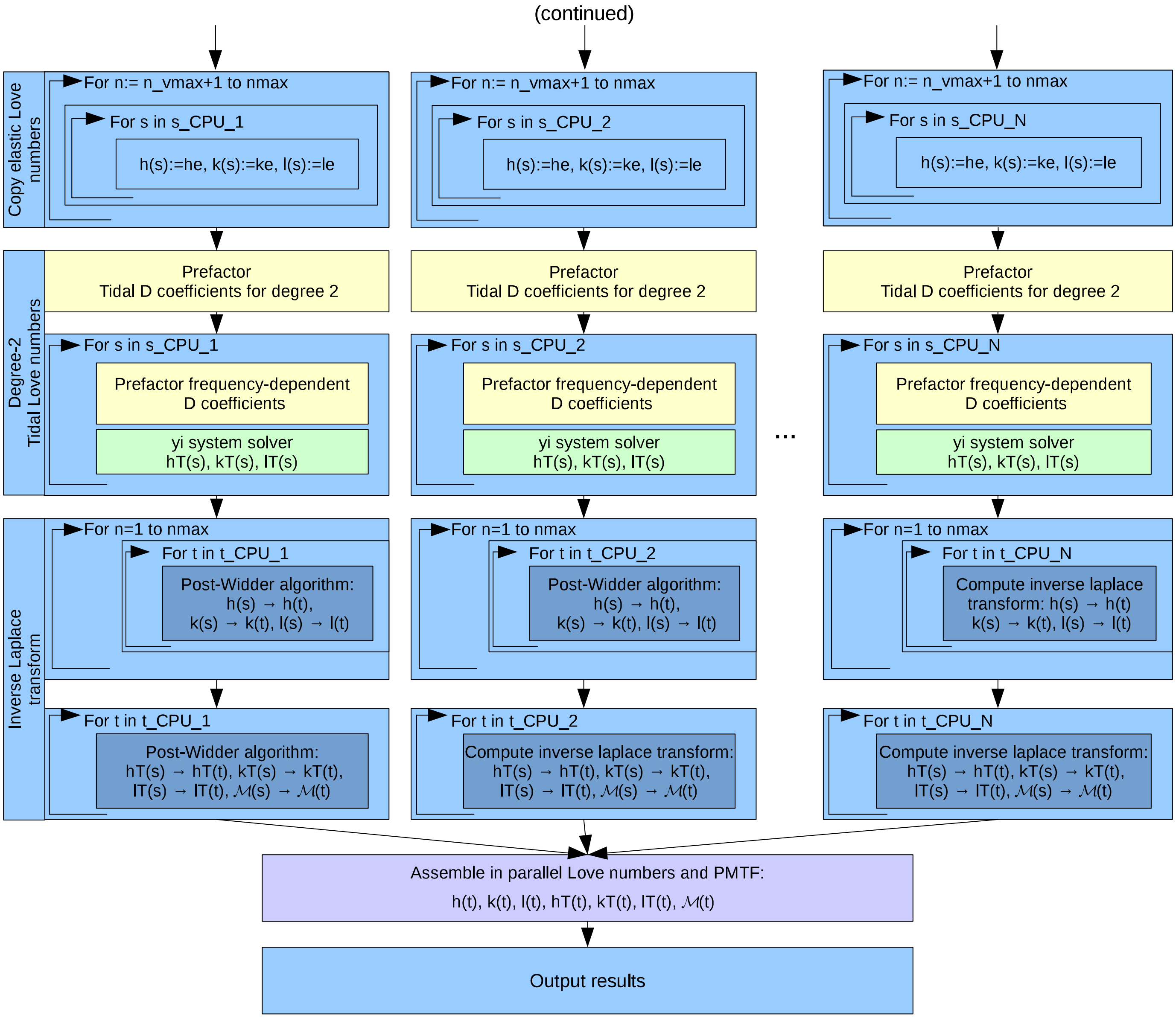

Figure S2. Flowchart of the Love number solver algorithm (continued).

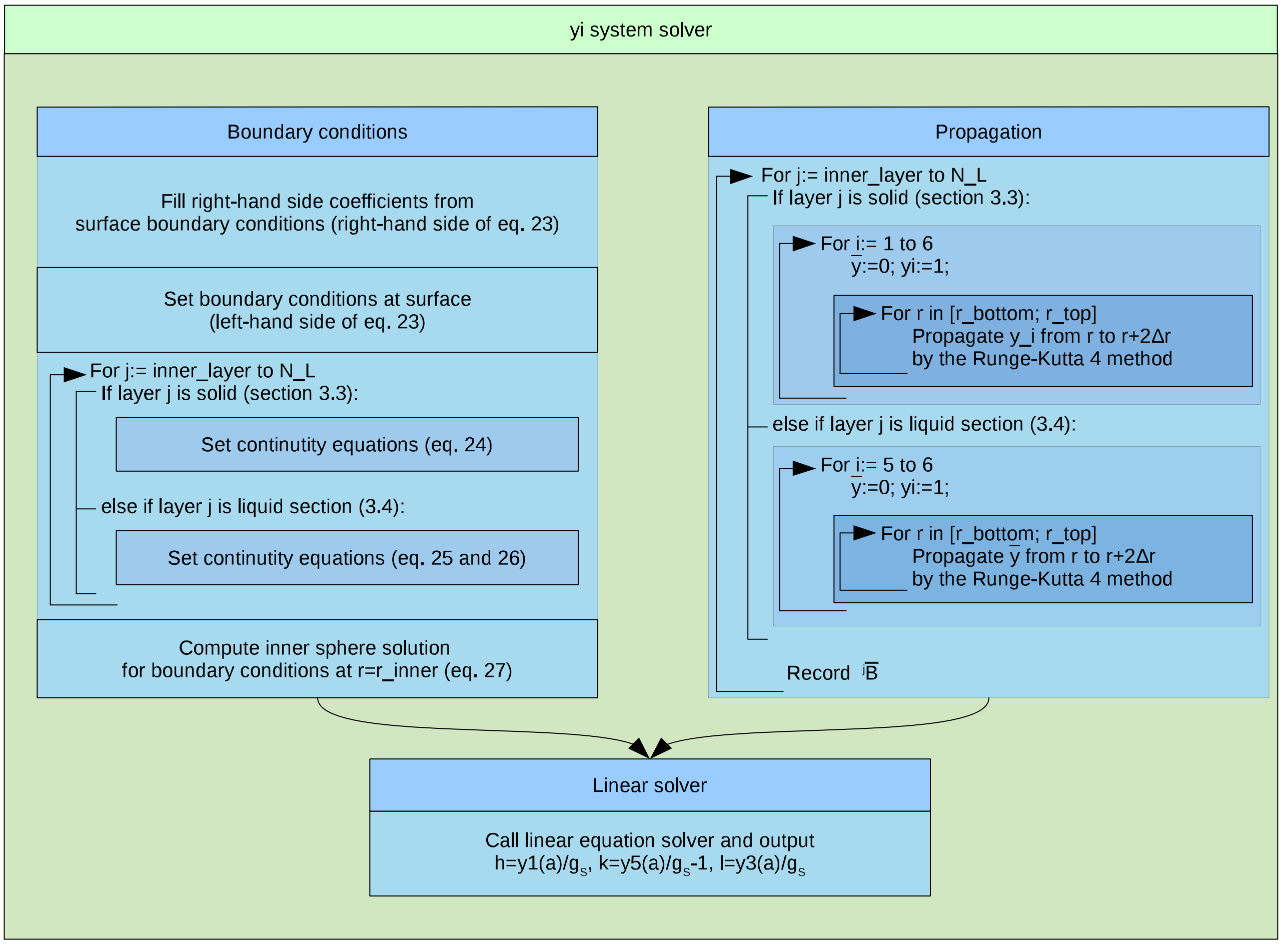

Figure S3. Flowchart of the y_i system solver algorithm for a given degree and frequency. Box headers indicate the categories in which computation times are reported in table 4.