S 1. Supplementary material to Lehsten et al. Simulating efficiently migration in dynamic vegetation models

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1 Derivation of the distribution of the SMSM kernel

The SMSM will shift the seed north, south, east or west with probability p, to north-west, north-east, south-west or south-east with probability $p/\sqrt{2}$. The discrete probability mass function is presented below

This shift can be seen as a two dimensional random variable (Y, X), where the 'north-south' Y component is independent of the 'east-west' X component. Furthermore, the distribution and hence all marginal moments are identical.

The expected value for X is zero, $E[X] = 0$, as the probability mass function is symmetric around 0. The same holds for the expected value for Y, $E[Y] = 0$.

The variance for X is given by $Var[X] = E[(X - E[X])^2]$. Straightforward calculations give that

$$var[X] = (p + 2p/\sqrt{2})(\Delta x)^2 + 0 + (p + 2p/\sqrt{2})(-\Delta x)^2 \tag{1}$$
$$= 2p(1 + \sqrt{2})(\Delta x)^2 \tag{2}$$

It will be convenient to define this quantity as

$$\sigma^2 = 2p(1 + \sqrt{2})(\Delta x)^2. \tag{3}$$

Again, $var[X] = var[Y]$ by symmetries. Finally, the covariance between X and Y is zero as the random variables are independent.

The SMSM iterates this kernel several times, say K iterations. Let Z be a column vector

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix} \tag{4}$$

The multivariate central limit theorem then states that

$$\lim_{k\to\infty} \frac{1}{\sqrt{K}} \sum_{k=1}^{K} Z_k - E[Z] \xrightarrow{d} MVN(0, \Sigma) \tag{5}$$

The interpretation of the multivariate central limit theorem is that a suitable scaled version of the sum of random variables converges in distribution, see Shiryaev [1996] to a multivariate Gaussian random variable.

Hence, we find that

$$\sum_{k=1}^{K} \begin{pmatrix} X \\ Y \end{pmatrix} \in MVN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, K \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \right). \tag{6}$$

Remark

This derivation holds on more general conditions as well. It is possible to derive the expected values, variances and covariances for any discrete probability mass function.

It is then possible to still derive central limit theorem results even if the random variable changes between the iterations. Formal conditions for the convergence of sums of independent but not identically distributed random variables are given for the Martingale central limit theorem, see Hall and Heyde [2014].

2 Derivation of the computational costs of the convolution

The computational complexity of the SMSM can be improved further if the kernel is separable, i.e. if the kernel can be written as a product of univariate terms. This is always possible for kernels that are joint densities for independent random variables, as by definition the joint density is the product of the marginal densities:

$$k_s(x, y) = k_s(x)k_s(y). \tag{7}$$

For separable kernels, it is possible to write the convolution:

$$S(x,y)**k(x,y) = S(x,y)**(k_s(x)**k_s(y)) = (S(x,y)**k(x))**k_s(y). \quad (8)$$

using the Helix transform where we used vector product of the marginal factors is identical to the convolution between these vectors. The last step was derived using the associative property of convolutions. The result is that the convolution can be computed in two steps, first the convolution between a matrix and a vector, leading to a computational cost of

O(N^2 R) followed by another convolution between a matrix and a vector, hence the overall cost will be

$$O(N^2K(R+R)). \quad (9)$$

References

Peter Hall and Christopher C Heyde. *Martingale limit theory and its application.* Academic press, 2014.

Albert N Shiryaev. *Probability, volume 95 of Graduate texts in mathematics.* Springer-Verlag, New York,, 1996.